Classifications of Cohen-Macaulay modules - The base ring associated to a transversal polymatroid

Ph.D. Thesis

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For my son Dorin-Andrei

Abstract

In this thesis, we focus on the study of the base rings associated to some transversal polymatroids. A transversal polymatroid is a special kind of discrete polymatroid. Discrete polymatroids were introduced by Herzog and Hibi [16] in 2002.

The thesis is structured in four chapters. Chapter 1 starts with a short excursion into convex geometry. Next, we look at some properties of affine semigroup rings. We recall some basic definitions and known facts about semigroup rings. Next we give a brief introduction to matroids and discrete polymatroids. Finally, we remind some properties of the base rings associated to discrete polymatroids. These properties will be needed in the next chapters of the thesis.

Chapter two is devoted to the study of the canonical module of the base ring associated to a transversal polymatroid. We determine the facets of the polyhedral cone generated by the exponent set of monomials defining the base ring. This allows us to describe the canonical module in terms of the relative interior of the cone. Also, this would allow one to compute the $a-$invariant of the base ring. Since the base ring associated to a discrete polymatroid is normal it follows that Ehrhart function is equal with Hilbert function and knowing the $a-$invariant we can very easy get its Hilbert series. We end this chapter with the following open problem

Open Problem: Let $n \geq 4$, $A_i \subset [n]$ for any $1 \leq i \leq n$ and $K[\mathcal{A}]$ be the base ring associated to the transversal polymatroid presented by $\mathcal{A} = \{A_1, \ldots, A_n\}$. If the Hilbert series is:

$$H_{K[\mathcal{A}]}(t) \;=\; \frac{1 + h_1\, t + \ldots + h_{n-r}\, t^{n-r}}{(1-t)^n},$$

then we have the following:
1) If $r = 1$, then $type(K[\mathcal{A}]) = 1 + h_{n-2} - h_1$.
2) If $2 \leq r \leq n$, then $type(K[\mathcal{A}]) = h_{n-r}$.

In chapter three we study intersections of Gorenstein base rings. These are also Gorenstein rings and we are interested when the intersections of Gorenstein base rings are the base rings associated to some transversal polymatroids. More precisely, we give necessary and sufficient conditions for the intersection of two base rings to be still a base ring of a transversal polymatroid.

In chapter four we study when the transversal polymatroids presented by $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ with $|A_i| = 2$ have the base ring $K[\mathcal{A}]$ Gorenstein. Using Worpitzky

identity, we prove that the numerator of the Hilbert series has the coefficients Eulerian numbers and from [1] the Hilbert series is unimodal.

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Contents.

Chapter 1

Background

1.1 A short excursion into convex geometry.

An *affine space* in $\mathbb{R}^n$ is a translation of a linear subspace of $\mathbb{R}^n$. Let $A \subset \mathbb{R}^n$ and $\mathit{aff}(A)$ be the affine space generated by A. Recall that $\mathit{aff}(A)$ is the set of all *affine combinations* of points in A:

$$\mathit{aff}(A) = \{a_1 p_1 + \ldots + a_r p_r \mid p_i \in A, a_1 + \ldots + a_r = 1, a_i \in \mathbb{R}\}.$$

There is a unique linear subspace V of $\mathbb{R}^n$ such that

$$\mathit{aff}(A) = x_0 + V,$$

for some $x_0 \in \mathbb{R}^n$. The dimension of $\mathit{aff}(A)$ is $\dim(\mathit{aff}(A)) = \dim_{\mathbb{R}}(V)$.

If $0 \neq a \in \mathbb{R}^n$, then H_a will denote the hyperplane of $\mathbb{R}^n$ through the origin with normal vector a, that is,

$$H_a = \{x \in \mathbb{R}^n \mid \langle x, a \rangle = 0\},$$

where $\langle, \rangle$ is the usual inner product in $\mathbb{R}^n$. The two closed halfspaces bounded by H_a are:

$$H_a^+ = \{x \in \mathbb{R}^n \mid \langle x, a \rangle \geq 0\} \text{ and } H_a^- = \{x \in \mathbb{R}^n \mid \langle x, a \rangle \leq 0\}.$$

Recall that a *polyhedral cone* $Q \subset \mathbb{R}^n$ is the intersection of a finite number of closed subspaces of the form H_a^+. If $Q = H_{a_1}^+ \cap \ldots \cap H_{a_m}^+$ is a polyhedral cone, then $\mathit{aff}(Q)$ is the

intersection of those hyperplanes H_{a_i}, $i = 1, \ldots, m$, that contain Q (see [4, Proposition 1.2.]). The *dimension of* Q is the dimension of $aff(Q)$, $\dim(Q) = \dim(aff(Q))$.

If $A = \{\gamma_1, \ldots, \gamma_r\}$ is a finite set of points in $\mathbb{R}^n$ the *cone* generated by A, denoted by $\mathbb{R}_+ A$, respectively the *convex hull* of A, denoted by $conv(A)$, are defined as

$$\mathbb{R}_+ A = \{\sum_{i=1}^{r} a_i \gamma_i \mid a_i \in \mathbb{R}_+ \text{ for all } 1 \leq i \leq n\}$$

respectively

$$conv(A) = \{\sum_{i=1}^{r} a_i \gamma_i \mid \sum_{i=1}^{r} a_i = 1, a_i \in \mathbb{R}_+ \text{ for all } 1 \leq i \leq n\},$$

where $\mathbb{R}_+$ denotes the set of nonnegative real numbers. An important fact is that Q is a polyhedral cone in $\mathbb{R}^n$ if and only if there exists a finite set $A \subset \mathbb{R}^n$ such that $Q = \mathbb{R}_+ A$ (see [4] or [35, Theorem 4.1.1.]). If U is a $\mathbb{Q}-$vector subspace of $\mathbb{R}^n$ such that $\dim_{\mathbb{Q}} U = n$, then we say that a cone is *rational* if is generated by a subset of U.

Next we give some important definitions and results (see [2], [3], [4], [27], [28]).

Definition 1.1.1. A *proper face* of a polyhedral cone Q is a subset $F \subset Q$ such that there is a supporting hyperplane H_a satisfying:

    1) $F = Q \cap H_a$,

    2) $Q \nsubseteq H_a$ and $Q \subset H_a^+$.

The *dimension* of a proper face F of a polyhedral cone Q is $\dim(F) = \dim(aff(F))$.

Definition 1.1.2. A cone C is *pointed* if 0 is a face of C. Equivalently we can require that $x \in C$ and $-x \in C \Rightarrow x = 0$.

Definition 1.1.3. The 1-dimensional faces of a pointed cone are called *extremal rays*.

Definition 1.1.4. A proper face F of a polyhedral cone $Q \subset \mathbb{R}^n$ is called a *facet* of Q if $\dim(F) = \dim(Q) - 1$.

Definition 1.1.5. If a polyhedral cone Q is written as

$$Q = H_{a_1}^+ \cap \ldots \cap H_{a_r}^+$$

such that no $H_{a_i}^+$ can be omitted, then we say that this is an *irreducible representation* of Q.

Theorem 1.1.6. *Let $Q \subset \mathbb{R}^n$, $Q \neq \mathbb{R}^n$, be a polyhedral cone with $\dim(Q) = n$. Then the halfspaces $H_{a_1}^+, \ldots, H_{a_m}^+$ in an irreducible representation $Q = H_{a_1}^+ \cap \ldots \cap H_{a_m}^+$ are uniquely determined. In fact, the sets $F_i = Q \cap H_{a_i}$, $i = 1, \ldots, n$, are the facets of Q.*

Proof. See [4, Theorem 1.6.] $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

The following two results are quite useful to determine the facets of a polyhedral cone.

Proposition 1.1.7. *Let A be a finite set of points in $\mathbb{Z}^n$. If F is a nonzero face of $\mathbb{R}_+ A$, then $F = \mathbb{R}_+ B$ for some $B \subset A$.*

Proof. Let $F = \mathbb{R}_+ A \cap H_a$ with $\mathbb{R}_+ A \subset H_a^+$. Then F is equal to the cone generated by the set $B = \{x \in A \mid \langle x, a \rangle = 0\}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Corollary 1.1.8. *Let A be a finite set of points in $\mathbb{Z}^n$ and F a face of $\mathbb{R}_+ A$.*
i) If $\dim F = 1$ and $A \subset \mathbb{N}^n$, then $F = \mathbb{R}_+ \alpha$ for some $\alpha \in A$.
ii) If $\dim \mathbb{R}_+ A = n$ and F is a facet defined by the supporting hyperplane H_a, then H_a is generated by a linearly independent subset of A.

Definition 1.1.9. Let Q be a polyhedral cone in $\mathbb{R}^n$ with $\dim Q = n$ and such that $Q \neq \mathbb{R}^n$. Let

$$Q = H_{a_1}^+ \cap \ldots \cap H_{a_r}^+$$

be the irreducible representation of Q. If $a_i = (a_{i1}, \ldots, a_{in})$, then we call

$$H_{a_i}(x) := a_{i1}x_1 + \ldots + a_{in}x_n = 0, \quad i \in [r],$$

the *equations of the cone Q.*

Definition 1.1.10. The *relative interior $ri(Q)$* of a polyhedral cone is the interior of Q with respect to the embedding of Q into its affine space $aff(Q)$, in which Q is full-dimensional.

The following result gives us the description of the relative interior of a polyhedral cone when we know its irreducible representation.

Theorem 1.1.11. *Let $Q \subset \mathbb{R}^n$, $Q \neq \mathbb{R}^n$, be a polyhedral cone with $\dim(Q) = n$ and let*

$$(*) \quad Q = H_{a_1}^+ \cap \ldots \cap H_{a_m}^+$$

be an irreducible representation of Q with $H_{a_1}^+, \ldots, H_{a_n}^+$ pairwise distinct, where $a_i \in \mathbb{R}^n \backslash \{0\}$ for all i. Set $F_i = Q \cap H_{a_i}$ for $i \in [r]$. Then:

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a) $ri(Q) = \{x \in \mathbb{R}^n \mid \langle x, a_1 \rangle > 0, \ldots, \langle x, a_r \rangle > 0\}$, where $ri(Q)$ is the relative interior of Q, which in this case is just the interior.

b) Each facet F of Q is of the form $F = F_i$ for some i.

c) Each F_i is a facet of Q.

Proof. See [2, Theorem 8.2.15] and [35, Theorem 3.2.1]. □

1.2 Affine semigroup rings.

An *affine semigroup* C is a finitely generated additive semigroup which for some $n \in \mathbb{N}$ is isomorphic to a subsemigroup of $\mathbb{Z}^n$ containing 0. For instance, $(\mathbb{N} \cup \{0\})^n$ is an affine semigroup for all $n \in \mathbb{N}$. Just like $\mathbb{Z}$ being generated by $\mathbb{N} \cup \{0\}$ as a group, for every affine semigroup C there exists a unique up to a canonical isomorphism finitely generated abelian group, denoted $\mathbb{Z}C$, that contains C and is generated by C as a group. Its rank is the *dimension* of C. Moreover, $\mathbb{R}C$ denotes $\mathbb{Z}C \otimes_{\mathbb{Z}} \mathbb{R}$, and one considers C to be a subset of $\mathbb{R}C$ via the canonical map $\mathbb{Z}C \to \mathbb{Z}C \otimes_{\mathbb{Z}} \mathbb{R}$. C is said to be *positive*, if 0 is the only invertible element in C.

Let C be an affine semigroup, and let K be a field. The $K-$ vector space

$$K[C] := \bigoplus_{a \in C} Kx^a$$

becomes a $K-$ algebra by setting $x^a \cdot x^b := x^{a+b}$ for all $a, b \in C$. It is the *affine semigroup ring* associated to C over K. We say that $K[C]$ is *positive* affine semigroup ring in case C is positive. Since C is a finitely generated semigroup, $K[C]$ is a finitely generated $K-$ algebra and thus Noetherian. Since $K[C]$ is a subring of $K[\mathbb{Z}C]$ and $K[\mathbb{Z}C]$ is isomorphic to the ring $K[x_1^{\pm}, \ldots, x_d^{\pm}]$ of Laurent polynomials, where d is the dimension of C, one obtains that $K[C]$ is an integral domain.

Assume that C is positive. A decomposition $R = \bigoplus_{n \geq 0} R_n$ of $R = K[C]$ is called an *admissible grading*, if the following conditions are fulfilled:

a) $R_0 = K$.

b) For all $n \geq 0$, R_n is a $K-$ vector space that is generated by finitely many elements of the form x^a, $a \in C$.

c) For all $m, n \geq 0$, $R_m \cdot R_n \subset R_{m+n}$.

The existence of an admissible grading is guaranteed by the following result.

Proposition 1.2.1. *If C is a positive affine semigroup, then there exist $r \in \mathbb{N}$ and a group homomorphism $\phi : \mathbb{Z}C \to \mathbb{Z}^r$, such that $\phi(C) \subseteq (\mathbb{N} \cup \{0\})^r$.*

Proof. See [3, Proposition 6.1.5]. □

An affine semigroup C is called *normal* if it satisfies the following condition: if $mz \in C$ for some $z \in \mathbb{Z}C$ and $m \in \mathbb{N}$, then $z \in C$. We have the following important results:

Proposition 1.2.2. (*Gordan's lemma*)
a) *If C is a normal semigroup, then $C = \mathbb{Z}C \cap \mathbb{R}_+ C$.*
b) *Let G be a finitely generated subgroup of $\mathbb{Q}^n$ and D a finitely generated rational cone in $\mathbb{R}^n$. Then $C = G \cap D$ is a normal semigroup.*

Theorem 1.2.3. *Let C be an affine semigroup, and let $K[C]$ be the associated affine semigroup ring over a field K. Then C is normal if and only if $K[C]$ is normal.*

Proof. One sees immediately that C must be normal if $K[C]$ is a normal domain: if x^z is an element of the field of fractions of $K[C]$ and if $(x^z)^m \in K[C]$ and $K[C]$ is normal, then $x^z \in K[C]$. For the proof of the converse, see [3, Theorem 6.1.4]. □

The following famous theorem is due to Hochster, see [3, Theorem 6.3.5] for a proof.

Theorem 1.2.4. *Let $R = K[C]$ be an affine semigroup ring. If R is normal, then it is Cohen-Macaulay.*

Let $R = K[x] = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K in the indeterminates $x_1, \ldots, x_n$ and $F = \{f_1, \ldots, f_q\}$ a finite set of distinct monomials in R such that $f_i \neq 1$ for all i. For $c \in \mathbb{N}^n$ we set $x^c = x_1^{c_1} \cdots x_n^{c_n}$. The *monomial subring* spanned by F is the $K-$subalgebra

$$K[F] = K[f_1, \ldots, f_q] \subset R.$$

The exponent vector of $f_i = x^{\alpha_i}$ is denoted by $\log(f_i) = \alpha_i$ and $\log(F)$ denotes the set of exponent vectors of the monomials in F.

Note that $K[F]$ is equal to the affine semigroup ring

$$K[C] = K[\{x^\alpha \mid \alpha \in C\}],$$

where $C = \mathbb{N}\log(f_1) + \ldots + \mathbb{N}\log(f_q)$ is the subsemigroup of $\mathbb{N}^n$ generated by $\log(F)$. Thus as $K-$ vector space $K[F]$ is generated by the set of monomials of the form x^α, with $\alpha \in C$. An important feature of $K[F]$ is that it is a graded subring of R with the standard grading $K[F]_i = K[F] \cap R_i$.

Next we give an important result of Danilov and Stanley which characterizes the canonical module in terms of the relative interior of a cone.

Theorem 1.2.5. (Danilov, Stanley) *Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and F a finite set of monomials in R. If $K[F]$ is normal, then the canonical module $\omega_{K[F]}$ of $K[F]$, with respect to standard grading, can be expressed as an ideal of $K[F]$ generated by monomials*

$$\omega_{K[F]} = (\{x^a \mid a \in \mathbb{N}A \cap ri(\mathbb{R}_+A)\}),$$

where $A = \log(F)$ and $ri(\mathbb{R}_+A)$ denotes the relative interior of $\mathbb{R}_+A$.

The formula above represents the canonical module of $K[F]$ as an ideal of $K[F]$ generated by monomials. For a comprehensive treatment of the Danilov-Stanley formula see [3], [27] or [28] .

1.3   Discrete polymatroids.

Matroid theory is one of the most fascinating research areas in combinatorics. The discrete polymatroid is a multiset analogue of the matroid. Based on the polyhedral theory on integral polymatroids developed in late 1960's and early 1970's, in the present section the combinatorics and algebra on discrete polymatroids will be studied.

Let $[n] = \{1, 2, \ldots, n\}$ and $2^{[n]}$ the set of all subsets of $[n]$. For a subset $A \subset [n]$ write $|A|$ for the cardinality of A. The following definition of the *matroid* is originated in Whitney (1935).

Definition 1.3.1. A *matroid* on the ground set $[n]$ is a nonempty subset $\mathcal{M} \subset 2^{[n]}$ satisfying:

(M_1) if $F_1 \in \mathcal{M}$ and $F_2 \subset F_1$, then $F_2 \in \mathcal{M}$;

(M_2) if $F_1, F_2 \in \mathcal{M}$ and $|F_1| < |F_2|$, then there is $x \in F_2 \setminus F_1$ such that $F_1 \cup \{x\} \in \mathcal{M}$.

The members of $\mathcal{M}$ are the *independent sets* of $\mathcal{M}$. A *base* of $\mathcal{M}$ is a maximal independent set of $\mathcal{M}$. It follows from (M_2) that if B_1 and B_2 are bases of $\mathcal{M}$, then $\mid B_1 \mid = \mid B_2 \mid$. The set of bases of $\mathcal{M}$ possesses the *"exchange property"* following:

(E) If B_1 and B_2 are bases of $\mathcal{M}$ and if $x \in B_1 \setminus B_2$, then there is $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\}$ is a base of $\mathcal{M}$.

Moreover, the set of bases of $\mathcal{M}$ possesses the following *"symmetric exchange property"*:

(SE) If B_1 and B_2 are bases of $\mathcal{M}$ and if $x \in B_1 \setminus B_2$, then there is $y \in B_2 \setminus B_1$ such that both $(B_1 \setminus \{x\}) \cup \{y\}$ and $(B_2 \setminus \{y\}) \cup \{x\}$ are bases of $\mathcal{M}$.

Alternatively, we can give another definition of matroid in terms of its set of bases. Given a nonempty set $\mathcal{B} \subset 2^{[n]}$, there exists a matroid $\mathcal{M}$ on the ground set $[n]$ with $\mathcal{B}$ its set of bases if and only if $\mathcal{B}$ possesses the exchange property (E). If we denote the canonical basis vectors of $\mathbb{R}^n$ by $e_1, e_2, \ldots, e_n$, then a matroid on $[n]$ can be regarded as a set of $(0,1)$−vectors $\sum_{k \in F} e_k$ for each $F \subset [n]$. Now we give three important examples of matroids.

Example 1.3.2. Vector Matroid: Let V be a vector space and E be a nonempty finite subset of V. We define the matroid $\mathcal{M}$ on the ground set E by taking the independent sets of $\mathcal{M}$ to be the sets of linearly independent elements in E. With linear algebra arguments one can check that the axioms of the matroid are fulfilled.

Cycle Matroid: Let G be a finite graph, with V its set of vertices and E its set of edges. Consider a set of edges independent if and only if it does not contain a simple cycle. Then the set of all these independent sets defines a matroid on the ground set E.

Uniform Matroid: Let r and n be nonnegative integers with r no larger than n. Let E be a set of cardinality n and let $\mathcal{M}$ be the collection of all subsets of E of cardinality r or less. Then $\mathcal{M}$ is a matroid, called the uniform matroid of rank r on n elements, and it is denoted by $U_{r,n}$.

Let $e_1, e_2, \ldots, e_n$ denote the canonical basis vectors of $\mathbb{R}^n$ and $\mathbb{R}^n_+$ denote the set of vectors $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ with each $u_i \geq 0$. Also, let $\mathbb{Z}^n_+ = \mathbb{R}^n_+ \cap \mathbb{Z}^n$. If $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are two vectors belonging to $\mathbb{R}^n_+$, then we write $u \leq v$ if all components

$v_i - u_i$ of $v - u$ are nonnegative, and write $u < v$ if $u \leq v$ and $u \neq v$. We say that u is a *subvector* of v if $u \leq v$. In addition, we set

$$u \vee v = (\max\{u_1, v_1\}, \ldots, \max\{u_n, v_n\}),$$

$$u \wedge v = (\min\{u_1, v_1\}, \ldots, \min\{u_n, v_n\}).$$

Thus, $u \wedge v \leq u \leq u \vee v$ and $u \wedge v \leq v \leq u \vee v$. The *modulus* of $u = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$ is $\mid u \mid = u_1 + \ldots + u_n$ and for a subset $F \subset [n]$, we set

$$u(F) = \sum_{k \in F} u_k.$$

Next we present the concept of *polymatroid* and its associated *rank function*. The concept of *polymatroid* originated in Edmonds ([10]), and for further properties the reader can consult ([14], [33]).

Definition 1.3.3. A *polymatroid* on the ground set $[n]$ is a nonempty compact subset $\mathcal{P} \subset \mathbb{R}_+^n$, the set of independent vectors, such that

$(\mathcal{P}_1)$ every subvector of an independent vector is independent;

$(\mathcal{P}_2)$ if $u, v \in \mathcal{P}$ with $\mid v \mid > \mid u \mid$, then there is a vector $w \in \mathcal{P}$ such that

$$u < w \leq u \vee v.$$

A *base* of a polymatroid $\mathcal{P} \subset \mathbb{R}_+^n$ is a maximal independent vector of $\mathcal{P}$, i.e. an independent vector $u \in \mathcal{P}$ with $u < v$ for no $v \in \mathcal{P}$. It follows from $(\mathcal{P}_2)$ that every base of $\mathcal{P}$ has the same modulus rank($\mathcal{P}$), the *rank* of $\mathcal{P}$.

Now we give an equivalent description of a polymatroid. Let $\mathcal{P} \subset \mathbb{R}_+^n$ be a polymatroid on the ground set $[n]$. The *ground set rank function* of $\mathcal{P}$ is a function $\rho : 2^{[n]} \to \mathbb{R}_+$ defined by setting

$$\rho(F) = \max\{v(F) \ : \ v \in \mathcal{P}\}$$

for all nonempty $F \in [n]$ together with $\rho(\emptyset) = 0$. Then we have

Proposition 1.3.4. ([33]) a) *Let $\mathcal{P} \subset \mathbb{R}_+^n$ be a polymatroid on the ground set $[n]$ and ρ its ground set rank function. Then ρ is nondecreasing, i.e., if $F_1 \subset F_2 \subset [n]$, then*

$\rho(F_1) \leq \rho(F_2)$, *and is submodular, i.e.,*

$$\rho(F_1) + \rho(F_2) \geq \rho(F_1 \cup F_2) + \rho(F_1 \cap F_2)$$

for all $F_1, F_2 \subset [n]$. Moreover, $\mathcal{P}$ coincides with the compact set

$$\{x \in \mathbb{R}^n_+ \mid x(A) \leq \rho(A), A \subset [n]\}.$$

b) Conversely, given a nondecresing and submodular function $\rho : 2^{[n]} \to \mathbb{R}_+$ with $\rho(\emptyset) = 0$, the compact set $\{x \in \mathbb{R}^n_+ \mid x(A) \leq \rho(A), A \subset [n]\}$ is a polymatroid on the ground set $[n]$ with ρ its ground set rank function.

From Proposition 1.3.4.a) it follows that a polymatroid $\mathcal{P} \subset \mathbb{R}^n_+$ on the ground set $[n]$ is a convex polytope in $\mathbb{R}^n_+$. A polymatroid is *integral* if and only if its ground set rank function is integer valued. For a detailed material on convex polytopes see [15], [31].

Now we introduce *discrete polymatroids*. They may be viewed as generalizations of matroids.

Definition 1.3.5. ([16]) Let $\mathcal{P}$ be a nonempty finite set of integer vectors in $\mathbb{R}^n_+$, which contains with each $u \in \mathcal{P}$ all its integral subvectors. The set $\mathcal{P}$ is called *discrete polymatroid* on the ground set $[n]$ if for all $u, v \in \mathcal{P}$ with $\mid v \mid > \mid u \mid$, there is a vector $w \in \mathcal{P}$ such that

$$u < w \leq u \vee v.$$

A *base* of $\mathcal{P}$ is a vector $u \in \mathcal{P}$ such that $u < v$ for no $v \in \mathcal{P}$. We denote by $B(\mathcal{P})$ the set of bases of a discrete polymatroid $\mathcal{P}$. It follows from the definition that any two bases of $\mathcal{P}$ have the same modulus. This common number is called the *rank* of $\mathcal{P}$.

Discrete polymatroids can be characterized in terms of their set of bases as follows.

Theorem 1.3.6. ([16]) *Let $\mathcal{P}$ be a nonempty finite set of integer vectors in $\mathbb{R}^n_+$, which contains with each $u \in \mathcal{P}$ all its integral subvectors, and let $B(\mathcal{P})$ be the set of vectors $u \in \mathcal{P}$ with $u < v$ for no $v \in \mathcal{P}$. The following conditions are equivalent:*
a) $\mathcal{P}$ is a discrete polymatroid;
b) if $u, v \in \mathcal{P}$ with $\mid v \mid > \mid u \mid$, there is an integer i such that $u + e_i \in \mathcal{P}$ and $u + e_i \leq u \vee v$;
c)

    i) all $u \in B(\mathcal{P})$ have the same modulus,

ii) if $u, v \in B(\mathcal{P})$ with $u_i > v_i$ there is $j \in [n]$ with $u_j < v_j$ such that $u - e_i + e_j \in B(\mathcal{P})$.

Condition $c).ii)$ from the theorem is also called the *exchange property*. An important consequence of this theorem is that it gives a way to construct discrete polymatroids. According to condition $c)$, it is enough to give a set of integer vector of the same modulus, which satisfy the *exchange property* and then, by taking all its integral subvectors, we obtain a discrete polymatroid.

The following result, which is obtained from Theorem 1.3.6. and the definition of matroid, shows that it makes sense to view the discrete polymatroids as generalizations of matroids.

Corollary 1.3.7. ([*16*]) *Let B be a nonempty finite set of integer vectors in $\mathbb{R}_+^n$. The following conditions are equivalent:*

   i) B is the set of bases of a matroid;

   ii) B is the set of bases of a discrete polymatroid, and for all $u \in B$ one has $u_k \leq 1$ for $k \in [n]$.

Just as in the case of matroids, we have the *symmetric exchange property*:

Theorem 1.3.8. ([*16*]) *If $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are bases of a discrete polymatroid $\mathcal{P} \subset \mathbb{Z}_+^n$, then for each $i \in [n]$ with $u_i > v_i$ there is $j \in [n]$ with $u_j < v_j$ such that both $u - e_i + e_j$ and $u - e_j + e_i$ are bases of $\mathcal{P}$.*

In the proof it is used the *rank function* of a discrete polymatroid, which we define next. Let $\mathcal{P} \subset \mathbb{Z}_+^n$ be a discrete polymatroid and $B(\mathcal{P})$ its set of bases. We define the *rank function* of the discrete polymatroid $\mathcal{P}$ to be function $\rho_{\mathcal{P}} : 2^{[n]} \to \mathbb{Z}_+$, by setting

$$\rho_{\mathcal{P}}(F) = \max\{u(F) \mid F \in \mathcal{P}\}$$

for all $\emptyset \neq F \subset [n]$, together with $\rho_{\mathcal{P}}(\emptyset) = 0$. It is clear that $\rho_{\mathcal{P}}$ is a nondecreasing function and from ([16]) we have that $\rho_{\mathcal{P}}$ is submodular. Conversely, given a nondecreasing and submodular function $\rho : 2^{[n]} \to \mathbb{Z}_+$, the set of $u \in \mathbb{Z}_+^n$ satisfying

$$u(F) \leq \rho(F), \text{ for all } F \in 2^{[n]}, \; (*)$$

is a discrete polymatroid, whose rank function is $\rho_{\mathcal{P}} = \rho$. In connection to the rank function ρ of a discrete polymatroid $\mathcal{P}$ we distinguish two important types of sets. A set $\emptyset \neq F \subset [n]$

is ρ-*closed* if any subset $G \subset [n]$ properly containing F satisfies $\rho(F) < \rho(G)$, and $\emptyset \neq F \subset [n]$ is ρ-*separable* if there exist two nonempty subsets F_1 and F_2 with $F_1 \cap F_2 = \emptyset$ and $F_1 \cup F_2 = F$ such that $\rho(F) = \rho(F_1) + \rho(F_2)$. A nonempty subset F in $[n]$ is ρ-*inseparable* if it is not ρ-*separable*. The following example is intended to give a better view to the construction $(*)$ and the definition above.

Example 1.3.9. ([29]) Let us consider the function $\rho_{\mathcal{P}} : 2^{[3]} \rightarrow \mathbb{Z}_+$ defined by $\rho(\emptyset) = 0$, $\rho(\{1\}) = 1$, $\rho(\{2\}) = 2$, $\rho(\{3\}) = 2$, $\rho(\{1,2\}) = 3$, $\rho(\{1,3\}) = 2$, $\rho(\{2,3\}) = 4$, $\rho(\{1,2,3\}) = 4$. One can easily check that ρ is nondecreasing and submodular. The bases are the integer solutions (u_1, u_2, u_3) of the inequations

$$u_1 \leq 1, u_2 \leq 2, u_3 \leq 2, u_1 + u_2 \leq 3, u_1 + u_3 \leq 2, u_2 + u_3 \leq 4,$$

together with

$$u_1 + u_2 + u_3 = 4,$$

i.e., the vectors $(1, 2, 1)$ and $(0, 2, 2)$. Taking all subintegral vectors of $(1, 2, 1)$ and $(0, 2, 2)$ we obtain the discrete polymatroid $\mathcal{P}$

$$\mathcal{P} = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (0,2,0), (0,0,2), (0,1,2),$$

$$(0,2,1), (1,1,1), (1,2,0), (0,2,2), (1,2,1)\}.$$

The $\rho - closed$ subsets of $[3]$ are: $\{1\}, \{2\}, \{1,2\}$ and $\{1,3\}$. The $\rho - inseparable$ subsets of $[3]$ are: $\{1\}, \{2\}, \{3\}$ and $\{1,3\}$.

The following result makes the connection between discrete polymatroids and integral polymatroids.

Theorem 1.3.10. ([16]) *A nonempty finite set $\mathcal{P} \subset \mathbb{Z}_+^n$ is a discrete polymatroid if and only if $conv(\mathcal{P}) \subset \mathbb{R}_+^n$ is an integral polymatroid with $conv(\mathcal{P}) \cap \mathbb{Z}_+^n = \mathcal{P}$.*

Now we present two techniques from [16] to construct discrete polymatroids. The first one shows that a nondecreasing and submodular function defined on a sublattice of $2^{[n]}$ produces a discrete polymatroid. A sublattice of $2^{[n]}$ is a collection $\mathcal{L}$ of subsets of $[n]$ with $\emptyset \in \mathcal{L}$ and $[n] \in \mathcal{L}$ such that for all $F, G \in \mathcal{L}$ both $F \cap G$ and $F \cup G$ belong to $\mathcal{L}$.

Theorem 1.3.11. ([*16*]) *Let $\mathcal{L}$ be a sublattice of $2^{[n]}$ and $\mu : \mathcal{L} \to \mathbb{R}_+$ an integer valued nondecreasing and submodular function with $\mu(\emptyset) = 0$. Then*

$$\mathcal{P}_{(\mathcal{L},\mu)} = \{u \in \mathbb{Z}^n_+ \mid u(F) \leq \mu(F), F \in \mathcal{L}\}$$

is a discrete polymatroid.

Example 1.3.12. ([16]) Let $\mathcal{L}$ be a chain of length n of $2^{[n]}$, say

$$\mathcal{L} = \{\emptyset, \{n\}, \{n-1, n\}, \ldots, \{1, \ldots, n\}\} \subset 2^{[n]}.$$

Given nonnegative integers $a_1, \ldots, a_n$, define $\mu : \mathcal{L} \to \mathbb{R}_+$ by

$$\mu(\{i, i+1, \ldots, n\}) = a_i + a_{i+1} + \ldots + a_n, \quad 1 \leq i \leq n,$$

together with $\mu(\emptyset) = 0$. Then the discrete polymatroid $\mathcal{P}_{(\mathcal{L},\mu)} \subset \mathbb{Z}^n_+$ is

$$\mathcal{P}_{(\mathcal{L},\mu)} = \{u \in \mathbb{Z}^n_+ \mid \sum_{k=i}^{n} u_k \leq \sum_{k=i}^{n} a_k, \quad 1 \leq k \leq n\}.$$

For the second result about construction of discrete polymatroids, first we need to fix some notation. Let $\mathcal{A} = \{A_1, \ldots, A_d\}$ be a family of nonempty subsets of $[n]$. It is not required that $A_i \neq A_j$ if $i \neq j$. Let

$$\mathcal{B}_\mathcal{A} = \{e_{i_1} + \ldots + e_{i_d} \mid i_k \in A_k, \ 1 \leq k \leq d\} \subset \mathbb{Z}^n_+$$

and define the integer valued nondecreasing function $\rho_\mathcal{A} : 2^{[n]} \to \mathbb{R}_+$ by setting

$$\rho_\mathcal{A}(X) = |\{k \mid A_k \cap X \neq \emptyset\}|, \ X \subset [n].$$

Now we can state

Theorem 1.3.13. ([*16*]) *The function $\rho_\mathcal{A}$ is submodular and $\mathcal{B}_\mathcal{A}$ is the set of bases of the discrete polymatroid $\mathcal{P}_\mathcal{A} \subset \mathbb{Z}^n_+$ arising from $\rho_\mathcal{A}$.*

The discrete polymatroid $\mathcal{P}_\mathcal{A} \subset \mathbb{Z}^n_+$ is called the *transversal polymatroid* presented by $\mathcal{A}$. The *rank* of $\mathcal{P}_\mathcal{A}$ is $rank(\mathcal{P}_\mathcal{A}) = d$. The following examples, given by Herzog and Hibi, show that the discrete polymatroid considered in Example 1.3.12. is a transversal polymatroid and that not all discrete polymatroids are transversal.

Example 1.3.14. ([16]) *a*) Let $r_1, \ldots, r_d \in [n]$ and set $A_k = [r_k]$, $1 \leq k \leq d$. Let $\min(X)$ denote the smallest integer belonging to X, where $\emptyset \neq X \subset [n]$. If $\mathcal{A} = \{A_1, \ldots, A_d\}$, then

$$\rho_{\mathcal{A}}(X) = \rho_{\mathcal{A}}(\{\min(X)\}) = |\{k \mid \min(X) \leq r_k\}|.$$

If $\emptyset \neq X \subset [n]$ is $\rho_{\mathcal{A}} - $ closed, then $X = \{\min(X), \min(X) + 1, \ldots, n)\}$. Let

$$a_i = |\{k \mid r_k = i\}|, \ 1 \leq i \leq n.$$

Thus

$$\rho_{\mathcal{A}}(\{i, i+1, \ldots, n\}) = a_i + a_{i+1} + \ldots + a_n, \ 1 \leq i \leq n.$$

Then the discrete polymatroid $\mathcal{P}_{\mathcal{A}} \subset \mathbb{Z}_+^n$ is

$$\mathcal{P}_{\mathcal{A}} = \{u \in \mathbb{Z}_+^n \mid \sum_{k=i}^{n} u_k \leq \sum_{k=i}^{n} a_k, \ 1 \leq k \leq n\}.$$

Thus $\mathcal{P}_{\mathcal{A}}$ coincides with the discrete polymatroid $\mathcal{P}_{(\mathcal{L},\mu)}$ in Example 1.3.12.

b) Let $\mathcal{P} \subset \mathbb{Z}_+^4$ denote the discrete polymatroid of rank 3 consisting of those $u \in \mathbb{Z}_+^4$ with $u_i \leq 2$ for $1 \leq i \leq 4$ and with $|u| \leq 3$. Then $\mathcal{P}$ is not transversal. Suppose, on the contrary, that $\mathcal{P}$ is a transversal polymatroid presented by $\mathcal{A} = \{A_1, A_2, A_3\}$ with each $A_k \subset [4]$. Since $(2,1,0,0), (2,0,1,0), (2,0,0,1) \in \mathcal{P}$ and $(3,0,0,0) \notin \mathcal{P}$ we may assume that $1 \in A_1, 1 \in A_2$ and $A_3 = \{2,3,4\}$. Since $(1,2,0,0), (0,2,1,0), (0,2,0,1) \in \mathcal{P}$ and $(0,3,0,0) \notin \mathcal{P}$ we may assume that $2 \in A_1$ and $A_2 = \{1,3,4\}$. Since $(0,0,2,1) \in \mathcal{P}$ and $(0,0,3,0) \notin \mathcal{P}$, one has $4 \in A_1$. Hence $(0,0,0,3) \in \mathcal{P}$, a contradiction.

1.4 The Ehrhart ring and the base ring of a discrete polymatroid.

Let K be a field and let $x_1, \ldots, x_n$ and s be indeterminates over K. If $u = (u_1, \ldots, u_n) \in \mathbb{Z}_+^n$, then we denote by x^u the monomial $x_1^{u_1} \cdots x_n^{u_n}$. Let P be a discrete polymatroid of rank d on the ground set $[n]$ with set of bases B. The toric ring $K[B]$ generated over K by the monomials x^u, where $u \in B$, is called the *base ring* of P. Since P is the set of integer vectors of an integral polymatroid $\mathcal{P}$ (see Theorem 1.3.10), we may study the *Ehrhart*

ring of $\mathcal{P}$. For this, one considers the cone $\mathcal{C} \subset \mathbb{R}^{n+1}$ with $\mathcal{C} = \mathbb{R}_+\{(p,1) \mid p \in \mathcal{P}\}$. Then $Q = \mathcal{C} \cap \mathbb{Z}^{n+1}$ is a subsemigroup of $\mathbb{Z}^{n+1}$, and the Ehrhart ring of $\mathcal{P}$ is defined to be the toric ring $K[\mathcal{P}] \subset K[x_1, \ldots, x_n, s]$ generated over K by the monomials $x^u s^i$, $(u,i) \in Q$. By Gordan's Lemma ([3], Proposition 6.1.2), $K[\mathcal{P}]$ is normal. Notice that $K[\mathcal{P}]$ is naturally graded if we assign to $x^u s^i$ the degree i. We denote by $K[P]$ the $K-$subalgebra of $K[\mathcal{P}]$ which is generated over K by the elements of degree 1 in $K[\mathcal{P}]$. Since $P = \mathcal{P} \cap \mathbb{Z}^n$ it follows that $K[P] = K[x^u s \mid u \in P]$. Observe that $K[B]$ may be identified with the subalgebra $K[x^u s \mid u \in B]$ of $K[P]$.

The base ring $K[B]$ was introduced in 1977 by N. White, in the particular case when B is the set of bases of a matroid, and he showed that for every matroid, the ring $K[B]$ is normal (see [34]) and thus Cohen-Macaulay. It is natural to ask whether the same holds for the base ring of any discrete polymatroid. Herzog and Hibi showed this.

Theorem 1.4.1. ([16]) $K[P] = K[\mathcal{P}]$. *In particular, $K[P]$ is normal.*

As corollary they also obtain

Corollary 1.4.2. ([16]) $K[B]$ *is normal.*

It is natural to ask, since both $K[P]$ and $K[B]$ are Cohen-Macaulay, when these rings are Gorenstein. Next we give some suggestive examples.

Example 1.4.3. ([16]) *a)* Let $P \subset \mathbb{Z}_+^3$ be the discrete polymatroid consisting of all integer vectors $u \in \mathbb{Z}_+^3$ with $|u| \leq 3$. Then the base ring $K[B]$ is the Veronese subring $K[x,y,z]^{(3)}$, which is Gorenstein. On the other hand, since the Hilbert series of the Ehrhart ring $K[P]$ is $(1 + 16t + 10t^2)/(1-t)^4$, it follows that $K[P]$ is not Gorenstein.
b) Let $P \subset \mathbb{Z}_+^3$ be the discrete polymatroid consisting of all integer vectors $u \in \mathbb{Z}_+^3$ with $|u| \leq 4$. Then the base ring $K[B] = K[x,y,z]^{(4)}$ is not Gorenstein. On the other hand, since the Hilbert series of the Ehrhart ring $K[P]$ is $(1 + 31t + 31t^2 + t^3)/(1-t)^4$, it follows that $K[P]$ is Gorenstein.
c) Let $P \subset \mathbb{Z}_+^2$ be the discrete polymatroid with $B = \{(1,2), (2,1)\}$ its set of bases. Then both $K[P]$ and $K[B]$ are Gorenstein.

However in [16] Herzog and Hibi give a combinatorial criterion for $K[P]$ to be Gorenstein.

Theorem 1.4.4. ([16]) *Let $P \subset \mathbb{Z}_+^n$ be a discrete polymatroid and suppose that the canonical basis vectors $e_1, \ldots, e_n$ of $\mathbb{R}^n$ belong to P. Let ρ denote the ground set rank function*

of the integral polymatroid $\mathcal{P} = conv(P) \subset \mathbb{R}^n$. Then the Ehrhart ring $K[P]$ of P is Gorenstein if and only if there exists an integer $\delta \geq 1$ such that

$$\rho(F) = \frac{1}{\delta}(|F| + 1)$$

for all $\rho - closed$ and $\rho - inseparable$ subsets F on $[n]$.

We now turn to the problem when the base ring of a discrete polymatroid is Gorenstein. A complete answer is not given so far. However, there are some particular classes for which there is a complete description. For example, in [17] there is a classification of the Gorenstein rings belonging to the class of algebras of Veronese type. Herzog and Hibi, in [16][Theorem 7.6.] find a characterization for the base ring of a generic discrete polymatroid to be Gorenstein.

We will end this chapter with the last phrase from the paper of Herzog and Hibi, *Discrete Polymatroids*:

″ It would, of course, be of interest to classify all transversal polymatroids with Gorenstein base rings. ″

Chapter 2

The type of the base ring associated to a transversal polymatroid

In this chapter we determine the facets of the polyhedral cone generated by the exponent set of the monomials defining the base ring associated to a transversal polymatroid presented by $\mathcal{A} = \{A_1, \ldots, A_n\}$. The importance of knowing those facets comes from the fact that the canonical module of the base ring can be expressed in terms of the relative interior of the cone. Since the base ring of a transversal polymatroid is normal using, Danilov-Stanley theorem we can find all minimal generators of the canonical module $\omega_{K[\mathcal{A}]}$ as an ideal of $K[\mathcal{A}]$ generated by monomials. So, we can compute the *type* of $K[\mathcal{A}]$. Also, this would allow us to compute the a-invariant of those base rings. The results presented were discovered by extensive computer algebra experiments performed with *Normaliz* [5].

2.1   Cones of dimension n with $n+1$ facets.

Let $n \in \mathbb{N}$, $n \geq 3$, $\sigma \in S_n$, $\sigma = (1, 2, \ldots, n)$ the cycle of length n, $[n] := \{1, 2, \ldots, n\}$ and $\{e_i\}_{1 \leq i \leq n}$ be the canonical base of $\mathbb{R}^n$. For a vector $x \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, we will denote $\mid x \mid := x_1 + \ldots + x_n$. If x^a is a monomial in $K[x_1, \ldots, x_n]$ we set $\log(x^a) = a$. Given a set A of monomials, the *log set of* A, denoted $\log(A)$, consists of all $\log(x^a)$ with $x^a \in A$.

We consider the following set of integer vectors of $\mathbb{N}^n$:

$$\downarrow i^{th} column$$

$$\nu^j_{\sigma^0[i]} := \Big(-j, \quad -j, \quad \ldots \quad , -j, \quad (n-j), \quad \ldots \quad , (n-j) \Big),$$

$$\downarrow (i+1)^{st} column$$

$$\nu^j_{\sigma^1[i]} := \Big((n-j), \quad -j, \quad \ldots \quad , -j, \quad (n-j), \quad \ldots \quad , (n-j) \Big),$$

$$\downarrow (i+2)^{nd} column$$

$$\nu^j_{\sigma^2[i]} := \Big((n-j), \quad (n-j) \quad , -j, \quad \ldots \quad , -j, \quad (n-j), \quad \ldots \quad , (n-j) \Big),$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$\downarrow (i-2)^{nd} column \qquad \downarrow (n-2)^{nd} column$$

$$\nu^j_{\sigma^{n-2}[i]} := \Big(-j, \quad \ldots \quad , -j, \quad (n-j), \quad \ldots \quad , (n-j), \quad -j, \quad -j \Big),$$

$$\downarrow (i-1)^{st} column \qquad \downarrow (n-1)^{st} column$$

$$\nu^j_{\sigma^{n-1}[i]} := \Big(-j, \quad \ldots \quad , -j, \quad (n-j), \quad \ldots \quad , (n-j), \quad -j \Big),$$

where $\sigma^k[i] := \{\sigma^k(1), \ldots, \sigma^k(i)\}$ for all $1 \leq i \leq n-2$, $1 \leq j \leq n-1$ and $0 \leq k \leq n-1$.

Remark : It is easy to see ([21]) that for any $1 \leq i \leq n-2$ and $0 \leq t \leq n-1$ we have $\nu^{n-i-1}_{\sigma^t[i]} = \nu_{\sigma^t[i]}$.

If A_i are some nonempty subsets of $[n]$ for $1 \leq i \leq m$, $\mathcal{A} = \{A_1, \ldots, A_m\}$, then the set of the vectors $\sum_{k=1}^m e_{i_k}$ with $i_k \in A_k$ is the base of a polymatroid, called the *transversal polymatroid presented* by $\mathcal{A}$. The *base ring* of a transversal polymatroid presented by $\mathcal{A}$ is the ring

$$K[\mathcal{A}] := K[x_{i_1} \cdots x_{i_m} \mid i_j \in A_j, 1 \leq j \leq m].$$

Lemma 2.1.1. *Let $n \in \mathbb{N}$, $n \geq 3$, $1 \leq i \leq n-2$ and $1 \leq j \leq n-1$. We consider the following two cases:*

a) If $i+j \leq n-1$, then let $A := \{\log(x_{j_1} \cdots x_{j_n}) \mid j_k \in A_k, \text{ for all } 1 \leq k \leq n\} \subset \mathbb{N}^n$ be the exponent set of generators of $K-$algebra $K[\mathcal{A}]$, where $\mathcal{A} = \{A_1 = [n], \ldots, A_i = [n], A_{i+1} = [n] \setminus [i], \ldots, A_{i+j} = [n] \setminus [i], A_{i+j+1} = [n], \ldots, A_n = [n]\}$.

b) If $i+j \geq n$, then let $A := \{\log(x_{j_1} \cdots x_{j_n}) \mid j_k \in A_k, \text{ for all } 1 \leq k \leq n\} \subset \mathbb{N}^n$ be the exponent set of generators of $K-$algebra $K[\mathcal{A}]$, where $\mathcal{A} = \{A_1 = [n] \setminus [i], \ldots, A_{i+j-n} = [n] \setminus [i], A_{i+j-n+1} = [n], \ldots, A_i = [n], A_{i+1} = [n] \setminus [i], \ldots, A_n = [n] \setminus [i]\}$.

Then the cone generated by A has the irreducible representation

$$\mathbb{R}_+ A = \bigcap_{a \in N} H_a^+,$$

where $N = \{\nu^j_{\sigma^0[i]}, \ e_k \mid 1 \leq k \leq n\}$ and $\{e_i\}_{1 \leq i \leq n}$ is the canonical base of $\mathbb{R}^n$.

Proof. We denote $J_k = \begin{cases} (n-j)\, e_k + j\, e_{i+1}, & \text{if } 1 \leq k \leq i \\ (n-j)\, e_1 + j\, e_k, & \text{if } i+2 \leq k \leq n \end{cases}$ and $J = n\, e_n$. Since $A_t = [n]$ for any $t \in \{1, \ldots, i\} \cup \{i+j+1, \ldots, n\}$ and $A_r = [n] \setminus [i]$ for any $r \in \{i+1, \ldots, i+j\}$ it is easy to see that for any $k \in \{1, \ldots, i\}$ and $r \in \{i+2, \ldots, n\}$ the set of monomials $x_k^{n-j}\, x_{i+1}^j$, $x_1^{n-j}\, x_r^j$, x_n^n is a subset of the generators of $K-$algebra $K[\mathcal{A}]$. Thus one has

$$\{J_1, \ldots, J_i, J_{i+2}, \ldots, J_n, J\} \subset A.$$

If we denote by C the matrix with the rows the coordinates of $\{J_1, \ldots, J_i, J_{i+2}, \ldots, J_n, J\}$, then by a simple computation we get $|\det (C)| = n\,(n-j)^i\, j^{n-i-1}$ for any $1 \leq i \leq n-2$ and $1 \leq j \leq n-1$. Thus, we get that the set

$$\{J_1, \ldots, J_i, J_{i+2}, \ldots, J_n, J\}$$

is linearly independent and it follows that dim $\mathbb{R}_+ A = n$. Since $\{J_1, \ldots, J_i, J_{i+2}, \ldots, J_n\}$ is linearly independent and lie on the hyperplane $H_{\nu^j_{\sigma^0[i]}}$ we have that $\dim(H_{\nu^j_{\sigma^0[i]}} \cap \mathbb{R}_+ A) = n-1$.

Now we will prove that $\mathbb{R}_+ A \subset H_a^+$ for all $a \in N$. It is enough to show that for all vectors $P \in A$, $\langle P, a \rangle \geq 0$ for all $a \in N$. Since $\{e_k\}_{1 \leq k \leq n}$ is the canonical base of $\mathbb{R}^n$, we get that $\langle P, e_k \rangle \geq 0$ for any $1 \leq k \leq n$. Let $P \in A$, $P = \log(x_{j_1} \cdots x_{j_n})$. We have two possibilities:

a) If $i+j \leq n-1$, then let t be the number of j_k such that $k \in \{1, \ldots, i\} \cup \{i+j+1, \ldots, n\}$ and $j_k \in [i]$. Thus $t \leq n-j$. Then $\langle P, \nu^j_{\sigma^0[i]} \rangle = -t\, j + (n-j-t)\,(n-j) + j(n-j) =$

$n(n - j - t) \geq 0$.

b) If $i + j \geq n$, then let t be the number of j_k such that $i + j - n + 1 \leq k \leq i$ and $j_k \in [i]$. Thus $t \leq n - j$. Then $\langle P, \nu^j_{\sigma^0[i]} \rangle = -t\, j + (n - j - t)\,(n - j) + j(n - j) = n(n - j - t) \geq 0$. Thus

$$\mathbb{R}_+ A \subseteq \bigcap_{a \in N} H_a^+.$$

Now we will prove the converse inclusion $\mathbb{R}_+ A \supseteq \bigcap_{a \in N} H_a^+$.

It is enough to prove that the extremal rays of the cone $\bigcap_{a \in N} H_a^+$ are in $\mathbb{R}_+ A$. Any extremal ray of the cone $\bigcap_{a \in N} H_a^+$ can be written as the intersection of $n - 1$ hyperplanes H_a, with $a \in N$. There are two possibilities to obtain extremal rays by intersection of $n - 1$ hyperplanes.

First case.

Let $1 \leq i_1 < \ldots < i_{n-1} \leq n$ be a sequence of integers and $\{t\} = [n] \setminus \{i_1, \ldots, i_{n-1}\}$. The

system of equations: $(*)$ $\begin{cases} z_{i_1} = 0, \\ \vdots \\ z_{i_{n-1}} = 0 \end{cases}$ admits the solution $x \in \mathbb{Z}_+^n$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ with

$|\,x\,| = n$, $x_k = n \cdot \delta_{kt}$ for all $1 \leq k \leq n$, where δ_{kt} is Kronecker's symbol.

There are two possibilities:

1) If $1 \leq t \leq i$, then $H_{\nu^j_{\sigma^0[i]}}(x) < 0$ and thus $x \notin \bigcap_{a \in N} H_a^+$.

2) If $i + 1 \leq t \leq n$, then $H_{\nu^j_{\sigma^0[i]}}(x) > 0$ and thus $x \in \bigcap_{a \in N} H_a^+$ and is an extremal ray.

Thus, there exist $n - i$ sequences $1 \leq i_1 < \ldots < i_{n-1} \leq n$ such that the system of equations $(*)$ has a solution $x \in \mathbb{Z}_+^n$ with $|\,x\,| = n$ and $H_{\nu^j_{\sigma^0[i]}}(x) > 0$.

The extremal rays are: $\{n\, e_k \mid i + 1 \leq k \leq n\}$.

Second case.

Let $1 \leq i_1 < \ldots < i_{n-2} \leq n$ be a sequence of integers and $\{r,\, s\} = [n] \setminus \{i_1, \ldots, i_{n-2}\}$, with $r < s$.

Let $x \in \mathbb{Z}_+^n$, with $|\,x\,| = n$, be a solution of the system of equations:

$(**)$ $\begin{cases} z_{i_1} = 0, \\ \vdots \\ z_{i_{n-2}} = 0, \\ -j\, z_1 - \ldots - j\, z_i + (n - j)z_{i+1} + \ldots + (n - j)z_n = 0. \end{cases}$

There are two possibilities:

1) If $1 \leq r \leq i$ and $i + 1 \leq s \leq n$, then the system of equations $(**)$ admits the solution

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{Z}_+^n, \ \text{with} \ \mid x \mid = n, \ \text{with} \ x_t = j \, \delta_{ts} + (n-j)\delta_{tr} \ \text{for all} \ 1 \leq t \leq n.$$

2) If $1 \leq r,s \leq i$ or $i+1 \leq r,s \leq n$, then there exists no solution $x \in \mathbb{Z}_+^n$ for the system of equations $(**)$ because otherwise $H_{\nu_{\sigma^0[i]}^j}(x) > 0$ or $H_{\nu_{\sigma^0[i]}^j}(x) < 0$.

Thus, there exist $i(n-i)$ sequences $1 \leq i_1 < \ldots < i_{n-2} \leq n$ such that the system of equations $(**)$ has a solution $x \in \mathbb{Z}_+^n$ with $\mid x \mid = n$, and the extremal rays are:

$$\{(n-j)e_r + j \, e_s \mid 1 \leq r \leq i \ \text{and} \ i+1 \leq s \leq n\}.$$

In conclusion, there exist $(i+1)(n-i)$ extremal rays of the cone $\bigcap_{a \in N} H_a^+$:

$$R := \{ne_k \mid i+1 \leq k \leq n\} \cup \{(n-j)e_r + j \, e_s \mid 1 \leq r \leq i \ \text{and} \ i+1 \leq s \leq n\}.$$

Since $A = \{\log(x_{j_1} \cdots x_{j_n}) \mid j_k \in A_k \in \mathcal{A}, \ \text{for all} \ 1 \leq k \leq n\} \subset \mathbb{N}^n$ where $\mathcal{A} = \{A_1 = [n], \ldots, A_i = [n], A_{i+1} = [n] \setminus [i], \ldots, A_{i+j} = [n] \setminus [i], A_{i+j+1} = [n], \ldots, A_n = [n]\}$ or $\mathcal{A} = \{A_1 = [n] \setminus [i], \ldots, A_{i+j-n} = [n] \setminus [i], A_{i+j-n+1} = [n], \ldots, A_i = [n], A_{i+1} = [n] \setminus [i], \ldots, A_n = [n] \setminus [i]\}$, it is easy to see that $R \subset A$. Thus, we have $\mathbb{R}_+ A = \bigcap_{a \in N} H_a^+$.

The representation is irreducible because if we delete, for some k, the hyperplane with the normal e_k, then the k'^{th} coordinate of $\log(x_{j_1} \cdots x_{j_n})$ would be negative, which is impossible; and if we delete the hyperplane with the normal $\nu_{\sigma^0[i]}^j$, then the cone $\mathbb{R}_+ A$ would be presented by $A = \{\log(x_{j_1} \cdots x_{j_n}) \mid j_k \in [n], \ \text{for all} \ 1 \leq k \leq n\}$ which is impossible. Thus the representation $\mathbb{R}_+ A = \bigcap_{a \in N} H_a^+$ is irreducible. $\qquad \square$

Lemma 2.1.2. *Let $n \in \mathbb{N}$, $n \geq 3$, $t \geq 1$, $1 \leq i \leq n-2$ and $1 \leq j \leq n-1$. Let $A = \{\log(x_{j_1} \cdots x_{j_n}) \mid j_{\sigma^t(k)} \in A_{\sigma^t(k)}, 1 \leq k \leq n\} \subset \mathbb{N}^n$ be the exponent set of generators of $K-algebra \ K[\mathcal{A}]$, where $\mathcal{A} = \{A_{\sigma^t(k)} \mid A_{\sigma^t(k)} = [n], \ for \ k \in [i] \cup ([n] \setminus [i+j]) \ and \ A_{\sigma^t(k)} = [n] \setminus \sigma^t[i], \ for \ i+1 \leq k \leq i+j\}, \ if \ i+j \leq n-1 \ or \ \mathcal{A} = \{A_{\sigma^t(k)} \mid A_{\sigma^t(k)} = [n] \setminus [i], \ for \ k \in [i+j-n] \cup ([n] \setminus [i]) \ and \ A_{\sigma^t(k)} = [n], \ for \ i+j-n+1 \leq k \leq i\}, \ if \ i+j \geq n. \ Then \ the cone generated by A has the irreducible representation*

$$\mathbb{R}_+ A = \bigcap_{a \in N} H_a^+,$$

where $N = \{\nu_{\sigma^t[i]}^j, \ e_k \mid 1 \leq k \leq n\}$.

Note that the algebras from Lemmas 2.1.1. and 2.1.2. are isomorphic.

Lemma 2.1.3. *a) Let $1 \le t \le n - i - j - 1$, $s \ge 2$ and $\beta \in \mathbb{N}^n$ be such that $H_{\nu^j_{\sigma^0[i]}}(\beta) = n(n-i-j-t)$ and $|\beta| = sn$. Then $\beta_1 + \ldots + \beta_i = (n-j)(s-1) + i + t$ and $\beta_{i+1} + \ldots + \beta_n = n + j(s-1) - i - t$.*
b) Let $r \ge 2$, $s \ge r$, $1 \le t \le r(n-j) - i$ and $\beta \in \mathbb{N}^n$ be such that $H_{\nu^j_{\sigma^0[i]}}(\beta) = nt$ and $|\beta| = sn$. Then $\beta_1 + \ldots + \beta_i = (n-j)s - t$ and $\beta_{i+1} + \ldots + \beta_n = js + t$.

Proof. a) Let $c = \beta_1 + \ldots + \beta_i$ so that $\beta_{i+1} + \ldots + \beta_n = sn - c$. Then $H_{\nu^j_{\sigma^0[i]}}(\beta) = -jc + (n-j)(sn - c) = n(sn - c - js)$. Since $H_{\nu^j_{\sigma^0[i]}}(\beta) = n(n-i-j-t)$, it follows that $sn - c - js = n - i - j - t$ and so $c = (n-j)(s-1) + i + t$. Thus, $\beta_1 + \ldots + \beta_i = (n-j)(s-1) + i + t$ and $\beta_{i+1} + \ldots + \beta_n = sn - c = n + j(s-1) - i - t$.
b) The proof goes as that for *a)*. $\qquad\square$

2.2 The type of base ring associated to transversal polymatroids with the cone of dimension n with $n + 1$ facets.

The main result of this chapter is the following.

Theorem 2.2.1. *Let $R = K[x_1, \ldots, x_n]$ be a standard graded polynomial ring over a field K and $\mathcal{A}$ satisfies the hypothesis of Lemma 2.1.1. Then:*
a) If $i + j \le n - 1$, then the type of $K[\mathcal{A}]$ is

$$type(K[\mathcal{A}]) = 1 + \sum_{t=1}^{n-i-j-1} \binom{n+i-j+t-1}{i-1} \binom{n-i+j-t-1}{n-i-1}.$$

b) If $i + j \ge n$, then the type of $K[\mathcal{A}]$ is

$$type(K[\mathcal{A}]) = \sum_{t=1}^{r(n-j)-i} \binom{r(n-j)-t-1}{i-1} \binom{rj+t-1}{n-i-1},$$

where $r = \left\lceil \frac{i+1}{n-j} \right\rceil$ ($\lceil x \rceil$ is the least integer $\ge x$).

Proof. Since $K[\mathcal{A}]$ is normal ([16]), the canonical module $\omega_{K[\mathcal{A}]}$ of $K[\mathcal{A}]$, with respect to standard grading, can be expressed as an ideal of $K[\mathcal{A}]$ generated by monomials

$$\omega_{K[\mathcal{A}]} = (\{x^a \mid a \in \mathbb{N}A \cap ri(\mathbb{R}_+A)\}),$$

where A is the exponent set of the $K-$ algebra $K[\mathcal{A}]$ and $ri(\mathbb{R}_+A)$ denotes the relative interior of $\mathbb{R}_+A$. By Lemma 2.1.1., the cone generated by A has the irreducible representation

$$\mathbb{R}_+A = \bigcap_{a \in N} H_a^+,$$

where $N = \{\nu_{\sigma^0[i]}^j,\ e_k \mid 1 \le k \le n\}$, $\{e_k\}_{1 \le k \le n}$ being the canonical base of $\mathbb{R}^n$.
a) Let $i \in [n-2]$, $j \in [n-1]$ be such that $i+j \le n-1$, $u_t = n+i-j+t$, $v_t = n-i+j-t$ for any $t \in [n-i-j-1]$. We will denote by M the set

$$M = \{\alpha \in \mathbb{N}^n \mid \alpha_k \ge 1,\ |(\alpha_1, \ldots, \alpha_i)| = u_t,\ |(\alpha_{i+1}, \ldots, \alpha_n)| = v_t \text{ for any } k \in [n],\ i \in [n-2],$$

$$j \in [n-1] \text{ such that } i+j \le n-1 \text{ and } t \in [n-i-j-1]\}.$$

We will show that

$$\mathbb{N}A \cap ri(\mathbb{R}_+A) = ((1, \ldots, 1) + (\mathbb{N}A \cap \mathbb{R}_+A)) \cup \bigcup_{\alpha \in M} (\{\alpha\} + (\mathbb{N}A \cap \mathbb{R}_+A)).$$

Since for any $\alpha \in M$, $H_{\nu_{\sigma^0[i]}^j}(\alpha) = -j(n+i-j+t)+(n-j)(n-i+j-t) = n(n-i-j-t) > 0$ and $H_{\nu_{\sigma^0[i]}^j}((1, \ldots, 1)) = n(n-i-j) > 0$, it follows that

$$\mathbb{N}A \cap ri(\mathbb{R}_+A) \supseteq ((1, \ldots, 1) + (\mathbb{N}A \cap \mathbb{R}_+A)) \cup \bigcup_{\alpha \in M} (\{\alpha\} + (\mathbb{N}A \cap \mathbb{R}_+A)).$$

Let $\beta \in \mathbb{N}A \cap ri(\mathbb{R}_+A)$, then $\beta_k \ge 1$ for any $k \in [n]$. Since $H_{\nu_{\sigma^0[i]}^j}((1, \ldots, 1)) = n(n-i-j) > 0$, it follows that $(1, \ldots, 1) \in ri(\mathbb{R}_+A)$. Let $\gamma \in \mathbb{N}^n$, $\gamma = \beta - (1, \ldots, 1)$. It is clear that $H_{\nu_{\sigma^0[i]}^j}(\gamma) = H_{\nu_{\sigma^0[i]}^j}(\beta) - n(n-i-j)$. If $H_{\nu_{\sigma^0[i]}^j}(\beta) \ge n(n-i-j)$, then $H_{\nu_{\sigma^0[i]}^j}(\gamma) \ge 0$. Thus $\gamma \in \mathbb{N}A \cap \mathbb{R}_+A$.
If $H_{\nu_{\sigma^0[i]}^j}(\beta) < n(n-i-j)$, then let $1 \le t \le n-i-j-1$ such that $H_{\nu_{\sigma^0[i]}^j}(\beta) = n(n-i-j-t)$.
   We claim that for any $\beta \in \mathbb{N}A \cap ri(\mathbb{R}_+A)$ with $|\beta| = sn \ge 2n$ and $t \in [n-i-j-1]$ such that $H_{\nu_{\sigma^0[i]}^j}(\beta) = n(n-i-j-t)$ we can find $\alpha \in M$ with $H_{\nu_{\sigma^0[i]}^j}(\alpha) = n(n-i-j-t)$ and $\beta - \alpha \in \mathbb{N}A \cap \mathbb{R}_+A$.

We proceed by induction on $s \geq 2$. If $s = 2$, then it is clear that $\beta \in M$. Indeed, then $\beta_1 + \ldots + \beta_i = n - a$, $\beta_{i+1} + \ldots + \beta_n = n + a$ for some $a \in \mathbb{Z}$ and so $H_{\nu^j_{\sigma^0[i]}}(\beta) = n(n - 2j + a) = n(n - i - j - t)$, that is $a = j - i - t$. It follows $\beta \in M$.

Suppose $s > 2$. Since $\beta_1 + \ldots + \beta_i = (n - j)(s - 1) + i + t \geq 2(n - j) + i + t$ by Lemma 2.1.3. we can get γ_e, with $0 \leq \gamma_e \leq \beta_e$ for any $1 \leq e \leq i$, such that $|(\gamma_1, \ldots, \gamma_i)| = n - j$. Since $\beta_{i+1} + \ldots + \beta_n = j(s - 1) + (n - i - t) \geq 2j + n - i - t$ by Lemma 2.1.3. we can get γ_e, with $0 \leq \gamma_e \leq \beta_e$ for any $i + 1 \leq e \leq n$, such that $|(\gamma_{i+1}, \ldots, \gamma_n)| = j$. It is clear that $\beta' = \beta - \gamma \in \mathbb{R}^n_+$,

$$| (\beta'_1, \ldots, \beta'_i) | = \sum_{e=1}^{i}(\beta_e - \gamma_e) = (n - j)(s - 2) + i + t,$$

$$| (\beta'_{i+1}, \ldots, \beta'_n) | = \sum_{e=i+1}^{n} (\beta_e - \gamma_e) = n + j(s - 2) - i - t$$

and

$$H_{\nu^j_{\sigma^0[i]}}(\beta') = H_{\nu^j_{\sigma^0[i]}}(\beta) - H_{\nu^j_{\sigma^0[i]}}(\gamma) = n(n - i - j - t) - [(-j)(n - j) + (n - j)j] = n(n - i - j - t).$$

So, $\beta' \in \mathbb{N}A \cap ri(\mathbb{R}_+A)$ and, since $|\beta'| = n(s - 1)$, we get by induction hypothesis that

$$\beta' \in \bigcup_{\alpha \in M} (\{\alpha\} + (\mathbb{N}A \cap \mathbb{R}_+A)).$$

Since $H_{\nu^j_{\sigma^0[i]}}(\gamma) = 0$ and $\gamma \in \mathbb{R}^n_+$, we get that $\gamma \in \mathbb{N}A \cap \mathbb{R}_+A$ and so

$$\beta \in \bigcup_{\alpha \in M} (\{\alpha\} + (\mathbb{N}A \cap \mathbb{R}_+A)).$$

Thus

$$\mathbb{N}A \cap ri(\mathbb{R}_+A) \subseteq ((1, \ldots, 1) + (\mathbb{N}A \cap \mathbb{R}_+A)) \cup \bigcup_{\alpha \in M} (\{\alpha\} + (\mathbb{N}A \cap \mathbb{R}_+A)).$$

So, the canonical module $\omega_{K[\mathcal{A}]}$ of $K[\mathcal{A}]$, with respect to standard grading, can be expressed as an ideal of $K[\mathcal{A}]$, generated by monomials

$$\omega_{K[\mathcal{A}]} = (\{x_1 \cdots x_n, \ x^\alpha | \ \alpha \in M\})K[\mathcal{A}].$$

The type of $K[\mathcal{A}]$ is the minimal number of generators of the canonical module. So, $type(K[\mathcal{A}]) = \#(M) + 1$, where $\#(M)$ is the cardinal of M.

Note that for any $t \in [n - i - j - 1]$, the equation $\alpha_1 + \ldots + \alpha_i = n + i - j + t$ has $\binom{n+i-j+t-1}{i-1}$ distinct nonnegative integer solutions with $\alpha_k \geq 1$, for any $k \in [i]$, respectively $\alpha_{i+1} + \ldots + \alpha_n = n - i + j - t$ has $\binom{n-i+j-t-1}{n-i-1}$ distinct nonnegative integer solutions with $\alpha_k \geq 1$ for any $k \in [n] \setminus [i]$. Thus, the cardinal of M is

$$\#(M) = \sum_{t=1}^{n-i-j-1} \binom{n+i-j+t-1}{i-1}\binom{n-i+j-t-1}{n-i-1}.$$

b) Let $i \in [n-2]$, $j \in [n-1]$ such that $i + j \geq n$, $u_t = r(n-j) - t$, $v_t = rj + t$ for any $t \in [r(n-j) - i]$, where $r = \left\lceil \frac{i+1}{n-j} \right\rceil$. We will denote by M' the set

$$M' = \{\alpha \in \mathbb{N}^n \mid \alpha_k \geq 1, |(\alpha_1, \ldots, \alpha_i)| = u_t, |(\alpha_{i+1}, \ldots, \alpha_n)| = v_t \text{ for any } k \in [n], i \in [n-2],$$

$$j \in [n-1] \text{ such that } i + j \geq n \text{ and } t \in [r(n-j)-i]\}.$$

We will show that

$$\mathbb{N}A \cap ri(\mathbb{R}_+A) = \bigcup_{\alpha \in M'}(\{\alpha\} + (\mathbb{N}A \cap \mathbb{R}_+A)).$$

Since for any $\alpha \in M'$, $H_{\nu^j_{\sigma^0_{[i]}}}(\alpha) = -j(r(n-j)-t) + (n-j)(rj+t) = nt > 0$, it follows that

$$\mathbb{N}A \cap ri(\mathbb{R}_+A) \supseteq \bigcup_{\alpha \in M'}(\{\alpha\} + (\mathbb{N}A \cap \mathbb{R}_+A)).$$

Let $\beta \in \mathbb{N}A \cap ri(\mathbb{R}_+A)$, then $\beta_k \geq 1$ for any $k \in [n]$. Since $H_{\nu^j_{\sigma^0_{[i]}}}((1,\ldots,1)) = n(n-i-j) < 0$, it follows that $(1,\ldots,1) \notin ri(\mathbb{R}_+A)$. We claim that $|\beta| \geq rn$. Indeed, since $\beta \in \mathbb{N}A \cap ri(\mathbb{R}_+A)$ and $|\beta| = sn$, it follows that

$$H_{\nu^j_{\sigma^0_{[i]}}}(\beta) = -j\sum_{k=1}^i \beta_i + (n-j)(sn - \sum_{k=1}^i \beta_i) > 0 \Longleftrightarrow \sum_{k=1}^i \beta_i < (n-j)s.$$

Hence $i + 1 \leq s(n-j)$ and so $r = \left\lceil \frac{i+1}{n-j} \right\rceil \leq s$.

We claim that for any $\beta \in \mathbb{N}A \cap ri(\mathbb{R}_+A)$ with $|\beta| = sn \geq rn$ and $t \in [r(n-j)-i]$ such that $H_{\nu^j_{\sigma^0_{[i]}}}(\beta) = nt$ we can find $\alpha \in M$ with $H_{\nu^j_{\sigma^0_{[i]}}}(\alpha) = nt$ such that $\beta - \alpha \in \mathbb{N}A \cap \mathbb{R}_+A$. We proceed by induction on $s \geq r$. If $s = r$, then it is clear that $\beta \in M'$. Indeed,

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then $\beta_1 + \ldots + \beta_i = n(r-1) - a$, $\beta_{i+1} + \ldots + \beta_n = n + a$ for some $a \in \mathbb{Z}$ and so $H_{\nu_{\sigma^0[i]}^j}(\beta) = n(n - jr + a) = nt$, that is $a = jr - n + t$. It follows $\beta \in M'$.

Suppose $s > r$. Since $\beta_1 + \ldots + \beta_i = (n-j)s - t \geq (n-j)(s-r) + i$, by Lemma 2.1.3 we can get γ_e, with $0 \leq \gamma_e \leq \beta_e$ for any $1 \leq e \leq i$, such that $\mid (\gamma_1, \ldots, \gamma_i) \mid = n - j$. Since $\beta_{i+1} + \ldots + \beta_n = js + t \geq jr + t$, by Lemma 2.1.3. we can get γ_e, with $0 \leq \gamma_e \leq \beta_e$ for any $i + 1 \leq e \leq n$, such that $\mid (\gamma_{i+1}, \ldots, \gamma_n) \mid = j$. It is clear that $\beta' = \beta - \gamma \in \mathbb{R}_+^n$,

$$\mid (\beta_1', \ldots, \beta_i') \mid = \sum_{e=1}^{i}(\beta_e - \gamma_e) = (n-j)(s-1) - t,$$

$$\mid (\beta_{i+1}', \ldots, \beta_n') \mid = \sum_{e=i+1}^{n}(\beta_e - \gamma_e) = j(s-1) + t$$

and
$$H_{\nu_{\sigma^0[i]}^j}(\beta') = H_{\nu_{\sigma^0[i]}^j}(\beta) - H_{\nu_{\sigma^0[i]}^j}(\gamma) = nt - [(-j)(n-j) + (n-j)j] = nt.$$

So, $\beta' \in \mathbb{N}A \cap ri(\mathbb{R}_+A)$ and, since $\mid \beta' \mid = n(s-1)$, we get by induction hypothesis that

$$\beta' \in \bigcup_{\alpha \in M'}(\{\alpha\} + (\mathbb{N}A \cap \mathbb{R}_+A)).$$

Since $H_{\nu_{\sigma^0[i]}^j}(\gamma) = 0$ and $\gamma \in \mathbb{R}_+^n$, we get that $\gamma \in \mathbb{N}A \cap \mathbb{R}_+A$ and so

$$\beta \in \bigcup_{\alpha \in M'}(\{\alpha\} + (\mathbb{N}A \cap \mathbb{R}_+A)).$$

Thus
$$\mathbb{N}A \cap ri(\mathbb{R}_+A) \subseteq \bigcup_{\alpha \in M'}(\{\alpha\} + (\mathbb{N}A \cap \mathbb{R}_+A)).$$

So, the canonical module $\omega_{K[\mathcal{A}]}$ of $K[\mathcal{A}]$, with respect to standard grading, can be expressed as an ideal of $K[\mathcal{A}]$, generated by monomials

$$\omega_{K[\mathcal{A}]} = (\{x^\alpha \mid \alpha \in M'\})K[\mathcal{A}].$$

The type of $K[\mathcal{A}]$ is the minimal number of generators of the canonical module. So, $type(K[\mathcal{A}]) = \#(M')$, where $\#(M')$ is the cardinal of M'.

Note that for any $t \in [r(n-j) - i]$, the equation $\alpha_1 + \ldots + \alpha_i = r(n-j) - t$ has

$\binom{r(n-j)-t-1}{i-1}$ distinct nonnegative integer solutions with $\alpha_k \geq 1$, for any $k \in [i]$, respectively $\alpha_{i+1} + \ldots + \alpha_n = rj + t$ has $\binom{rj+t-1}{n-i-1}$ distinct nonnegative integer solutions with $\alpha_k \geq 1$ for any $k \in [n] \setminus [i]$.

Thus, the cardinal of M' is

$$\#(M') = \sum_{t=1}^{r(n-j)-i} \binom{r(n-j)-t-1}{i-1}\binom{rj+t-1}{n-i-1}.$$

$\square$

Corollary 2.2.2. $K[\mathcal{A}]$ *is Gorenstein ring if and only if* $i + j = n - 1$.

Proof. " $\Leftarrow$ " From Theorem 2.2.1.a) $type(K[\mathcal{A}]) = 1$. See also [21], Lemma 5.1.

" $\Rightarrow$ " If $i + j \leq n - 1$, since $n + i - j + t - 1 \geq n + i - j \geq n + 1 - j \geq 2$ and $n - i + j - t - 1 \geq n - i + j - 1 - (n - i - j - 1) = 2(j - 1) \geq 0$, it follows that $type(K[\mathcal{A}]) = 1 \Leftrightarrow n - i - j - 1 = 0$ (q.e.d.) or $n - i - j - 1 \geq 1$ and $n - i + j - t - 1 < n - i - 1$ for $t \in [n - i - j] \Rightarrow j < 1$, which is false.

If $i + j \geq n$, then using Theorem 2.2.1.b) we have $type(K[\mathcal{A}]) = 1 \Leftrightarrow r(n - j) - i = 1$, $r(n - j) - 2 = i - 1$ and $rj = n - i - 1 \Rightarrow rn = n \Leftrightarrow 1 = \left\lceil \frac{i+1}{n-j} \right\rceil \geq \frac{i+1}{n-j} \geq \frac{n-j+1}{n-j} > 1$, which is false. $\square$

Let S be a standard graded $K-$algebra over a field K. Recall that the $a-invariant$ of S, denoted $a(S)$, is the degree as a rational function of the Hilbert series of S, see for instance ([28, p. 99]). If S is Cohen-Macaulay and ω_S is the canonical module of S, then

$$a(S) = -\min\{i \mid (\omega_S)_i \neq 0\},$$

see [3, p. 141] and [28, Proposition 4.2.3]. In our situation $S = K[\mathcal{A}]$ is normal [16] and consequently Cohen-Macaulay, thus this formula applies. We have the following consequence of Theorem 2.2.1.

Corollary 2.2.3. *The $a-invariant$ of $K[\mathcal{A}]$ is* $a(K[\mathcal{A}]) = \begin{cases} -1, & if \quad i + j \leq n - 1, \\ -r, & if \quad i + j \geq n. \end{cases}$

Proof. Let $\{x^{\alpha_1}, \ldots, x^{\alpha_q}\}$ be generators of the $K-$algebra $K[\mathcal{A}]$. Then $K[\mathcal{A}]$ is a standard graded algebra with the grading

$$K[\mathcal{A}]_i = \sum_{|c|=i} K(x^{\alpha_1})^{c_1} \cdots (x^{\alpha_q})^{c_q}, \text{ where } |c| = c_1 + \ldots + c_q.$$

If $i + j \leq n - 1$, then $\min\{i \mid (\omega_{K[\mathcal{A}]})_i \neq 0\} = 1$ and so we have $a(K[\mathcal{A}]) = -1$.

If $i + j \geq n$, then $\min\{i \mid (\omega_{K[\mathcal{A}]})_i \neq 0\} = r$. So, we have $a(K[\mathcal{A}]) = -r$. $\qquad\square$

2.3 Ehrhart function

We consider a fixed set of distinct monomials $F = \{x^{\alpha_1}, \ldots, x^{\alpha_r}\}$ in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K.

Let
$$\mathcal{P} = conv(\log(F))$$

be the convex hull of the set $\log(F) = \{\alpha_1, \ldots, \alpha_r\}$. The *normalized Ehrhart ring* of $\mathcal{P}$ is the graded algebra
$$A_{\mathcal{P}} = \bigoplus_{i=0}^{\infty} (A_{\mathcal{P}})_i \subset R[T],$$

where the $i - $th component is given by
$$(A_{\mathcal{P}})_i = \sum_{\alpha \in \mathbb{Z} \ \log(F) \cap \ i\mathcal{P}} K \ x^{\alpha} \ T^i.$$

The *normalized Ehrhart function* of $\mathcal{P}$ is defined as
$$E_{\mathcal{P}}(t) = \dim_K (A_{\mathcal{P}})_t = \mid \mathbb{Z} \log(F) \cap \ t \mathcal{P} \mid.$$

An important result of [28], Corollary 7.2.45 is the following.

Theorem 2.3.1. *If $K[F]$ is a standard graded subalgebra of R and h is the Hilbert function of $K[F]$, then:*

a) $h(t) \leq E_{\mathcal{P}}(t)$ *for all $t \geq 0$, and*

b) $h(t) = E_{\mathcal{P}}(t)$ *for all $t \geq 0$ if and only if $K[F]$ is normal.*

In this section we will compute the Hilbert function and the Hilbert series for the $K-$ algebra $K[\mathcal{A}]$, where $\mathcal{A}$ satisfies the hypothesis of Lemma 2.1.1.

Proposition 2.3.2. *In the hypothesis of Lemma 2.1.1, the Hilbert function of the $K-$ algebra $K[\mathcal{A}]$ is*
$$h(t) = \sum_{k=0}^{(n-j)t} \binom{k+i-1}{k} \binom{nt - k + n - i - 1}{nt - k}.$$

Proof. From [16] we know that the $K-$ algebra $K[\mathcal{A}]$ is normal. Thus, to compute the Hilbert function of $K[\mathcal{A}]$ is equivalent to compute the Ehrhart function of $\mathcal{P}$, where $\mathcal{P} = conv(A)$ (see Theorem 2.3.1.).

It is clear that $\mathcal{P} = \{x \in \mathbb{R}^n \mid x_i \geq 0, \ 0 \leq x_1 + \ldots + x_i \leq n - j \text{ and } x_1 + \ldots + x_n = n\}$ and thus $t \ \mathcal{P} = \{x \in \mathbb{R}^n \mid x_i \geq 0, \ 0 \leq x_1 + \ldots + x_i \leq (n-j) \ t \text{ and } x_1 + \ldots + x_n = n \ t\}$.

Since for any $0 \leq k \leq (n-j) \ t$ the equation $x_1 + \ldots + x_i = k$ has $\binom{k+i-1}{k}$ nonnegative integer solutions and the equation $x_{i+1} + \ldots + x_n = n \ t - k$ has $\binom{nt-k+n-i-1}{nt-k}$ nonnegative integer solutions, we get that

$$E_\mathcal{P}(t) = \mid \mathbb{Z} \ A \ \cap \ t \ \mathcal{P} \mid = \sum_{k=0}^{(n-j)t} \binom{k+i-1}{k} \binom{nt-k+n-i-1}{nt-k}.$$

$\square$

Corollary 2.3.3. *In the hypothesis of Lemma* 2.1.1.*, the Hilbert series of the $K-$ algebra $K[\mathcal{A}]$ is*

$$H_{K[\mathcal{A}]}(t) \ = \ \frac{1 + h_1 \ t + \ldots + h_{n-r} \ t^{n-r}}{(1-t)^n},$$

where

$$h_j = \sum_{s=0}^{j}(-1)^s \ h(j-s) \ \binom{n}{s},$$

$h(s)$ *being the Hilbert function and* $r = \left\lceil \frac{i+1}{n-j} \right\rceil$.

Proof. Since the $a-$invariant of $K[\mathcal{A}]$ is $a(K[\mathcal{A}]) = -r$, with $r = \left\lceil \frac{i+1}{n-j} \right\rceil$, it follows that to compute the Hilbert series of $K[\mathcal{A}]$ is necessary to know the first $n - r + 1$ values of the Hilbert function of $K[\mathcal{A}]$, $h(i)$ for $0 \leq i \leq n - r$. Since $\dim(K[\mathcal{A}]) = n$, applying n times the difference operator Δ (see [3]) on the Hilbert function of $K[\mathcal{A}]$ we get the conclusion.

Let $\Delta^0(h)_j := h(j)$ for any $0 \leq j \leq n - r$. For $k \geq 1$ let $\Delta^k(h)_0 := 1$ and $\Delta^k(h)_j := \Delta^{k-1}(h)_j - \Delta^{k-1}(h)_{j-1}$ for any $1 \leq j \leq n - r$. We claim that

$$\Delta^k(h)_j = \sum_{s=0}^{k}(-1)^s h(j-s)\binom{k}{s}$$

for any $k \geq 1$ and $0 \leq j \leq n - r$. We proceed by induction on k.

If $k = 1$, then

$$\Delta^1(h)_j = \Delta^0(h)_j - \Delta^0(h)_{j-1} = h(j) - h(j-1) = \sum_{s=0}^{1}(-1)^s h(j-s)\binom{1}{s}$$

for any $1 \leq j \leq n - r$.

If $k > 1$, then

$$\Delta^k(h)_j = \Delta^{k-1}(h)_j - \Delta^{k-1}(h)_{j-1} = \sum_{s=0}^{k-1}(-1)^s h(j-s)\binom{k-1}{s} - \sum_{s=0}^{k-1}(-1)^s h(j-1-s)\binom{k-1}{s}$$

$$= h(j)\binom{k-1}{0} + \sum_{s=1}^{k-1}(-1)^s h(j-s)\binom{k-1}{s} - \sum_{s=0}^{k-2}(-1)^s h(j-1-s)\binom{k-1}{s} + (-1)^k h(j-k)\binom{k-1}{k-1}$$

$$= h(j) + \sum_{s=1}^{k-1}(-1)^s h(j-s)\left[\binom{k-1}{s} + \binom{k-1}{s-1}\right] + (-1)^k h(j-k)\binom{k-1}{k-1}$$

$$= h(j) + \sum_{s=1}^{k-1}(-1)^s h(j-s)\binom{k}{s} + (-1)^k h(j-k)\binom{k-1}{k-1} = \sum_{s=0}^{k}(-1)^s h(j-s)\binom{k}{s}.$$

Thus, if $k = n$ it follows that

$$h_j = \Delta^n(h)_j = \sum_{s=0}^{n}(-1)^s h(j-s)\binom{n}{s} = \sum_{s=0}^{j}(-1)^s h(j-s)\binom{n}{s}$$

for any $1 \leq j \leq n - r$.

$\square$

Next we will give some examples.

Example 2.3.4. Let $\mathcal{A} = \{A_1, \ldots, A_7\}$, where $A_1 = A_2 = A_3 = A_6 = A_7 = [7]$, $A_4 = A_5 = [7] \setminus [3]$. The cone genereated by A, the exponent set of generators of $K-$algebra $K[\mathcal{A}]$, has the irreducible representation

$$\mathbb{R}_+ A = H^+_{\nu^2_{\sigma^0[3]}} \cap H^+_{e_1} \cap \ldots \cap H^+_{e_7}.$$

The type of $K[\mathcal{A}]$ is

$$type(K[\mathcal{A}]) = 1 + \binom{8}{2}\binom{4}{3} = 113.$$

The Hilbert series of $K[\mathcal{A}]$ is

$$H_{K[\mathcal{A}]}(t) = \frac{1 + 1561t + 24795t^2 + 57023t^3 + 25571t^4 + 1673t^5 + t^6}{(1-t)^7}.$$

Note that $type(K[\mathcal{A}]) = 1 + h_5 - h_1 = 113$.

Example 2.3.5. Let $\mathcal{A} = \{A_1, \ldots, A_7\}$, where $A_3 = A_4 = [7]$, $A_1 = A_2 = A_5 = A_6 = A_7 = [7] \setminus [4]$. The cone genereated by A, the exponent set of generators of $K-$algebra $K[\mathcal{A}]$, has the irreducible representation

$$\mathbb{R}_+ A = H^+_{\nu^5_{\sigma^0[4]}} \cap H^+_{e_1} \cap \ldots \cap H^+_{e_7}.$$

The type of $K[\mathcal{A}]$ is

$$type(K[\mathcal{A}]) = \binom{4}{3}\binom{15}{2} + \binom{3}{3}\binom{16}{2} = 540.$$

The Hilbert series of $K[\mathcal{A}]$ is

$$H_{K[\mathcal{A}]}(t) = \frac{1 + 351t + 2835t^2 + 3297t^3 + 540t^4}{(1-t)^7}.$$

Note that $type(K[\mathcal{A}]) = h_4 = 540$.

We end this section with the following open problem.

Open Problem: Let $n \geq 4$, $A_i \subset [n]$ for any $1 \leq i \leq n$ and $K[\mathcal{A}]$ be the base ring associated to the transversal polymatroid presented by $\mathcal{A} = \{A_1, \ldots, A_n\}$. If the Hilbert series is:

$$H_{K[\mathcal{A}]}(t) = \frac{1 + h_1\, t + \ldots + h_{n-r}\, t^{n-r}}{(1-t)^n},$$

then we have the following:

1) If $r = 1$, then $type(K[\mathcal{A}]) = 1 + h_{n-2} - h_1$.
2) If $2 \leq r \leq n$, then $type(K[\mathcal{A}]) = h_{n-r}$.

Chapter 3

Intersections of base rings associated to transversal polymatroids

The discrete polymatroids and their base rings are studied recently in many papers (see [6], [16], [18], [29], [30]). It is important to give conditions when the base ring associated to a transversal polymatroid is Gorenstein (see [16]). In [21] we introduced a class of such base rings. In this paper we note that an intersection of such base rings (introduced in [21]) is Gorenstein and give necessary and sufficient conditions for the intersection of two base rings from [21] to be still a base ring of a transversal polymatroid. Also, we compute the a-invariant of those base rings. The results presented were discovered by extensive computer algebra experiments performed with *Normaliz* [5].

3.1 Intersection of cones of dimension n with $n+1$ facets.

Let $r \geq 2$, $1 \leq i_1, \ldots, i_r \leq n-2$, $0 = t_1 \leq t_2, \ldots, t_r \leq n-1$ and consider r presentations of transversal polymatroids:

$$\mathcal{A}_s = \{A_{s,k} \mid A_{s,\sigma^{t_2}(k)} = [n], \text{ if } k \in [i_2] \cup \{n\},\ A_{s,\sigma^{t_2}(k)} = [n] \setminus \sigma^{t_2}[i_2], \text{ if } k \in [n-1] \setminus [i_2]\}$$

for any $1 \leq s \leq r$. From [21] we know that the base rings $K[\mathcal{A}_s]$ are Gorenstein rings

and the cones generated by the exponent vectors of the monomials defining the base ring associated to a transversal polymatroid presented by $\mathcal{A}_s$ are

$$\mathbb{R}_+ A_s = \bigcap_{a \in N_s} H_a^+,$$

where $N_s = \{\nu_{\sigma^{t_s}[i_s]}, \nu_{\sigma^k[n-1]} \mid 0 \leq k \leq n-1\}$, $A_s = \{\log(x_{j_1} \cdots x_{j_n}) \mid j_k \in A_{s,k}, 1 \leq k \leq n\} \subset \mathbb{N}^n$

for any $1 \leq s \leq r$.

We denote by $K[A_1 \cap \ldots \cap A_r]$, the $K-$algebra generated by x^α with $\alpha \in A_1 \cap \ldots \cap A_r$.
It is clear that one has

$$\mathbb{R}_+(A_1 \cap \ldots \cap A_r) \subseteq \mathbb{R}_+ A_1 \cap \ldots \cap \mathbb{R}_+ A_r = \bigcap_{a \in N_1 \cup \ldots \cup N_r} H_a^+.$$

Conversely, since

$$A_1 \cap \ldots \cap A_r = \{\alpha \in \mathbb{N}^n \mid \; \mid \alpha \mid = n, \; H_{\nu_{\sigma^{t_s}[i_s]}}(\alpha) \geq 0, \text{ for any } 1 \leq s \leq r\},$$

we have that

$$\mathbb{R}_+(A_1 \cap \ldots \cap A_r) \supseteq \bigcap_{a \in N_1 \cup \ldots \cup N_r} H_a^+$$

and so

$$\mathbb{R}_+(A_1 \cap \ldots \cap A_r) = \bigcap_{a \in N_1 \cup \ldots \cup N_r} H_a^+.$$

We claim that the intersection

$$\bigcap_{a \in N_1 \cup \ldots \cup N_r} H_a^+$$

is the irreducible representation of the cone $\mathbb{R}_+(A_1 \cap \ldots \cap A_r)$.
We prove by induction on $r \geq 1$. If $r = 1$, then the intersection

$$\bigcap_{a \in N_1} H_a^+$$

is the irreducible representation of the cone $\mathbb{R}_+(A_1)$ (see [21], Lemma 4.1).
If $r > 1$ then we have two cases to study:
1) if we delete, for some $0 \leq k \leq n - 1$, the hyperplane with the normal $\nu_{\sigma^k[n-1]}$, then a coordinate of a $\log(x_{j_1} \cdots x_{j_{i_1}} x_{j_{i_1}+1} \cdots x_{j_{n-1}} x_{j_n})$ would be negative, which is impossible;

2) if we delete, for some $1 \leq s \leq r$, the hyperplane with the normal $\nu_{\sigma^{t_s}[i_s]}$, then

$$\bigcap_{a \in N_1 \cup \ldots \cup N_{s-1} \cup N_{s+1} \ldots N_r} H_a^+$$

is by induction the irreducible representation of the cone $\mathbb{R}_+(A_1 \cap \ldots \cap A_{s-1} \cap A_{s+1} \ldots \cap A_r)$, which is different from $\mathbb{R}_+(A_1 \cap \ldots \cap A_r)$. Hence, the intersection

$$\bigcap_{a \in N_1 \cup \ldots \cup N_r} H_a^+$$

is the irreducible representation of the cone $\mathbb{R}_+(A_1 \cap \ldots \cap A_r)$.

Lemma 3.1.1. *The* $K-$ *algebra* $K[A_1 \cap \ldots \cap A_r]$ *is a Gorenstein ring.*

Proof. We will show that the canonical module $\omega_{K[A_1 \cap \ldots \cap A_r]}$ is generated by $(x_1 \cdots x_n) K[A_1 \cap \ldots \cap A_r]$. Since the semigroups $\mathbb{N}(A_t)$ are normal for any $1 \leq t \leq r$, it follows that $\mathbb{N}(A_1 \cap \ldots \cap A_r)$ is normal. Then the $K-$ algebra $K[A_1 \cap \ldots \cap A_r]$ is normal (see [3] Theorem 6.1.4. p. 260) and using the Danilov-Stanley theorem we get that the canonical module $\omega_{K[A_1 \cap \ldots \cap A_r]}$ is

$$\omega_{K[A_1 \cap \ldots \cap A_r]} = (\{x^\alpha \mid \alpha \in \mathbb{N}(A_1 \cap \ldots \cap A_r) \cap ri(\mathbb{R}_+(A_1 \cap \ldots \cap A_r))\}).$$

Let d_t be the greatest common divisor of n and $i_t + 1$, $\gcd(n, i_t + 1) = d_t$, for any $1 \leq t \leq r$. For any $1 \leq s \leq r$, there exist two possibilities for the equation of the facet $H_{\nu_{\sigma^{t_s}[i_s]}}$:

1) If $i_s + t_s \leq n$, then the equation of the facet $H_{\nu_{\sigma^{t_s}[i_s]}}$ is

$$H_{\nu_{\sigma^{t_s}[i_s]}}(y): \quad \frac{(i_s + 1)}{d_s} \sum_{k=1}^{t_s} y_k - \frac{(n - i_s - 1)}{d_s} \sum_{k=t_s+1}^{t_s+i_s} y_k + \frac{(i_s + 1)}{d_s} \sum_{k=t_s+i_s+1}^{n} y_k = 0.$$

2) If $i_s + t_s > n$, then the equation of the facet $H_{\nu_{\sigma^{t_s}[i_s]}}$ is

$$H_{\nu_{\sigma^{t_s}[i_s]}}(y): \quad -\frac{(n - i_s - 1)}{d_s} \sum_{k=1}^{i_s+t_s-n} y_k + \frac{(i_s + 1)}{d_s} \sum_{k=i_s+t_s-n+1}^{t_s} y_k - \frac{(n - i_s - 1)}{d_s} \sum_{k=t_s+1}^{n} y_k = 0.$$

The relative interior of the cone $\mathbb{R}_+(A_1 \cap \ldots \cap A_r)$ is

$$ri(\mathbb{R}_+(A_1 \cap \ldots \cap A_r)) = \{y \in \mathbb{R}^n \mid y_k > 0, \ H_{\nu_{\sigma^{t_s}[i_s]}}(y) > 0 \text{ for any } 1 \leq k \leq n \text{ and } 1 \leq s \leq r\}.$$

We will show that

$$\mathbb{N}(A_1\cap\ldots\cap A_r)\cap ri(\mathbb{R}_+(A_1\cap\ldots\cap A_r)) = (1,\ldots,1) + (\mathbb{N}(A_1\cap\ldots\cap A_r)\cap\mathbb{R}_+(A_1\cap\ldots\cap A_r)).$$

It is clear that $ri(\mathbb{R}_+(A_1\cap\ldots\cap A_r)) \supset (1,\ldots,1) + \mathbb{R}_+(A_1\cap\ldots\cap A_r)$.
If $(\alpha_1,\alpha_2,\ldots,\alpha_n)\in\mathbb{N}(A_1\cap\ldots\cap A_r)\cap ri(\mathbb{R}_+(A_1\cap\ldots\cap A_r))$, then $\alpha_k\geq 1$ for any $1\leq k\leq n$ and for any $1\leq s\leq r$ we have

$$\frac{(i_s+1)}{d_s}\sum_{k=1}^{t_s}\alpha_k - \frac{(n-i_s-1)}{d_s}\sum_{k=t_s+1}^{t_s+i_s}\alpha_k + \frac{(i_s+1)}{d_s}\sum_{k=t_s+i_s+1}^{n}\alpha_k \geq 1, \text{ if } i_s+t_s\leq n$$

or

$$-\frac{(n-i_s-1)}{d_s}\sum_{k=1}^{i_s+t_s-n}\alpha_k + \frac{(i_s+1)}{d_s}\sum_{k=i_s+t_s-n+1}^{t_s}\alpha_k - \frac{(n-i_s-1)}{d_s}\sum_{k=t_s+1}^{n}\alpha_k \geq 1, \text{ if } i_s+t_s > n$$

and

$$\sum_{k=1}^{n}\alpha_k = t\,n \text{ for some } t\geq 1.$$

We claim that there exists $(\beta_1,\beta_2,\ldots,\beta_n)\in\mathbb{N}(A_1\cap\ldots\cap A_r)\cap\mathbb{R}_+(A_1\cap\ldots\cap A_r)$ such that $(\alpha_1,\alpha_2,\ldots,\alpha_n) = (\beta_1+1,\beta_2+1,\ldots,\beta_n+1)$. Let $\beta_k=\alpha_k-1$ for all $1\leq k\leq n$.
It is clear that $\beta_k\geq 0$ and for any $1\leq s\leq r$,

$$H_{\nu_{\sigma^{t_s}[i_s]}}(\beta) = H_{\nu_{\sigma^{t_s}[i_s]}}(\alpha) - H_{\nu_{\sigma^{t_s}[i_s]}}(1,\ldots,1) = H_{\nu_{\sigma^{t_s}[i_s]}}(\alpha) - \frac{n}{d_s}.$$

If $H_{\nu_{\sigma^{t_s}[i_s]}}(\beta) = j_s$, for some $1\leq s\leq r$ and $1\leq j_s\leq\frac{n}{d_s}-1$, then we will get a contradiction. Indeed, since n divides $\sum_{k=1}^{n}\alpha_k$, it follows that $\frac{n}{d_s}$ divides j_s, which is false.
Hence, we have $(\beta_1,\beta_2,\ldots,\beta_n)\in\mathbb{N}(A_1\cap\ldots\cap A_r)\cap\mathbb{R}_+(A_1\cap\ldots\cap A_r)$ and $(\alpha_1,\alpha_2,\ldots,\alpha_n)\in\mathbb{N}(A_1\cap\ldots\cap A_r)\cap ri(\mathbb{R}_+(A_1\cap\ldots\cap A_r))$.
Since $\mathbb{N}(A_1\cap\ldots\cap A_r)\cap ri(\mathbb{R}_+(A_1\cap\ldots\cap A_r)) = (1,\ldots,1) + (\mathbb{N}(A_1\cap\ldots\cap A_r)\cap\mathbb{R}_+(A_1\cap\ldots\cap A_r))$, we get that $\omega_{K[A_1\cap\ldots\cap A_r]} = (x_1\cdots x_n)K[A_1\cap\ldots\cap A_r]$. $\qquad\square$

Let S be a standard graded $K-$algebra over a field K. Recall that the $a-$invariant of S, denoted $a(S)$, is the degree as a rational function of the Hilbert series of S, see for instance ([28], p. 99). If S is Cohen-Macaulay and ω_S is the canonical module of S, then

$$a(S) = -\min\{i\mid(\omega_S)_i\neq 0\},$$

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see ([3], p. 141) and ([28], Proposition 4.2.3). In our situation $S = K[A_1 \cap \ldots \cap A_r]$ is normal and consequently Cohen-Macaulay, thus this formula applies. As consequence of Lemma 3.1.1. we have the following.

Corollary 3.1.2. *The $a - invariant$ of $K[A_1 \cap \ldots \cap A_r]$ is $a(K[A_1 \cap \ldots \cap A_r]) = -1$.*

Proof. Let $\{x^{\alpha_1}, \ldots, x^{\alpha_q}\}$ be the generators of the $K - $algebra $K[A_1 \cap \ldots \cap A_r]$. $K[A_1 \cap \ldots \cap A_r]$ is a standard graded algebra with the grading

$$K[A_1 \cap \ldots \cap A_r]_i = \sum_{|c|=i} K(x^{\alpha_1})^{c_1} \cdots (x^{\alpha_q})^{c_q}, \text{ where } |c| = c_1 + \ldots + c_q.$$

Since $\omega_{K[A_1 \cap \ldots \cap A_r]} = (x_1 \cdots x_n) K[A_1 \cap \ldots \cap A_r]$, it follows that $\min \{i \mid (\omega_{K[A_1 \cap \ldots \cap A_r]})_i \neq 0\} = 1$ and so $a(K[A_1 \cap \ldots \cap A_r]) = -1$. $\square$

3.2 When is $K[A \cap B]$ the base ring associated to some transversal polymatroid?

Let $n \geq 2$ and consider two transversal polymatroids presented by $\mathcal{A} = \{A_1, \ldots, A_n\}$, respectively $\mathcal{B} = \{B_1, \ldots, B_n\}$. Let A and B be the set of exponent vectors of monomials defining the base rings $K[\mathcal{A}]$, respectively $K[\mathcal{B}]$, and $K[A \cap B]$ the $K - $algebra generated by x^α with $\alpha \in A \cap B$.

Question: There exists a transversal polymatroid such that its base ring is the $K-$algebra $K[A \cap B]$?

In the following we will give two suggestive examples.

Example 1. Let $n = 4$, $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$, $\mathcal{B} = \{B_1, B_2, B_3, B_4\}$, where $A_1 = A_4 = B_2 = B_3 = \{1, 2, 3, 4\}$, $A_2 = A_3 = \{2, 3, 4\}$, $B_1 = B_4 = \{1, 3, 4\}$ and $K[\mathcal{A}]$, $K[\mathcal{B}]$ the base rings associated to transversal polymatroids presented by $\mathcal{A}$, respectively $\mathcal{B}$. It is easy to see that the generators set of $K[\mathcal{A}]$, respectively $K[\mathcal{B}]$, is given by $A = \{y \in \mathbb{N}^4 \mid |y| = 4, \ 0 \leq y_1 \leq 2, \ y_k \geq 0, \ 1 \leq k \leq 4\}$, respectively $B = \{y \in \mathbb{N}^4 \mid |y| = 4, \ 0 \leq y_2 \leq 2, \ y_k \geq 0, \ 1 \leq k \leq 4\}$. We show that the $K-$ algebra $K[A \cap B]$ is the base ring of the transversal polymatroid presented by $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$, where $C_1 = C_4 = \{1, 3, 4\}$, $C_2 = C_3 = \{2, 3, 4\}$.

Since the base ring associated to the transversal polymatroid presented by $\mathcal{C}$ has the exponent set $C = \{y \in \mathbb{N}^4 \mid |y| = 4, \ 0 \leq y_1 \leq 2, \ 0 \leq y_2 \leq 2, \ y_k \geq 0, \ 1 \leq k \leq 4\}$, it follows that $K[A \cap B] = K[\mathcal{C}]$. Thus, in this example $K[A \cap B]$ is the base ring of a transversal polymatroid.

Example 2. Let $n = 4$, $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$, $\mathcal{B} = \{B_1, B_2, B_3, B_4\}$ where $A_1 = A_2 = A_4 = B_1 = B_2 = B_3 = \{1,2,3,4\}$, $A_3 = \{3,4\}$, $B_4 = \{1,4\}$ and $K[\mathcal{A}]$, $K[\mathcal{B}]$ the base rings associated to the transversal polymatroids presented by $\mathcal{A}$, respectively $\mathcal{B}$. It is easy to see that the generators set of $K[\mathcal{A}]$, respectively $K[\mathcal{B}]$, is $A = \{y \in \mathbb{N}^4 \mid |y| = 4, \ 0 \leq y_1 + y_2 \leq 3, \ y_k \geq 0, \ 1 \leq k \leq 4\}$, respectively $B = \{y \in \mathbb{N}^4 \mid |y| = 4, \ 0 \leq y_2 + y_3 \leq 3, \ y_k \geq 0, \ 1 \leq k \leq 4\}$. We claim that there exists no transversal polymatroid $\mathcal{P}$ such that the $K-$ algebra $K[A \cap B]$ is its base ring. Suppose, on the contrary, that $\mathcal{P}$ is presented by $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$ with each $C_k \subset [4]$. Since $(3, 0, 1, 0), (3, 0, 0, 1) \in \mathcal{P}$ and $(3, 1, 0, 0) \notin \mathcal{P}$, we may assume by changing the numerotation of $\{C_i\}_{i=\overline{1,4}}$ that $1 \in C_1, 1 \in C_2, 1 \in C_4$ and $C_3 = \{3, 4\}$. Since $(0, 3, 0, 1) \in \mathcal{P}$, we may assume that $2 \in C_1, 2 \in C_2, 2 \in C_4$. Hence $(0, 3, 1, 0) \in \mathcal{P}$, a contradiction.

Let $1 \leq i_1, i_2 \leq n - 2$, $0 \leq t_2 \leq n - 1$ and $\tau \in S_{n-2}$, $\tau = (1, 2, \ldots, n - 2)$ the cycle of length $n - 2$. We consider two transversal polymatroids presented by:

$$\mathcal{A} = \{A_k \mid A_k = [n], \text{ if } k \in [i_1] \cup \{n\}, \ A_k = [n] \setminus [i_1], \text{ if } k \in [n - 1] \setminus [i_1]\}$$

and

$$\mathcal{B} = \{B_k \mid B_{\sigma^{t_2}(k)} = [n], \text{ if } k \in [i_2] \cup \{n\}, \ B_{\sigma^{t_2}(k)} = [n] \setminus \sigma^{t_2}[i_2], \text{ if } k \in [n - 1] \setminus [i_2]\}$$

such that A, respectively B, is the exponent vectors of the monomials defining the base ring associated to the transversal polymatroid presented by $\mathcal{A}$, respectively $\mathcal{B}$. From [21] we know that the base rings $K[\mathcal{A}]$ and $K[\mathcal{B}]$ are Gorenstein rings and the cones generated by the exponent vectors of the monomials defining the base rings associated to the transversal polymatroids presented by $\mathcal{A}$ and $\mathcal{B}$ are:

$$\mathbb{R}_+ A = \bigcap_{a \in N_1} H_a^+, \quad \mathbb{R}_+ B = \bigcap_{a \in N_2} H_a^+,$$

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where $N_1 = \{\nu_{\sigma^0[i_1]}, \; \nu_{\sigma^k[n-1]} \mid 0 \le k \le n-1\}, \quad N_2 = \{\nu_{\sigma^{t_2}[i_2]}, \; \nu_{\sigma^k[n-1]} \mid 0 \le k \le n-1\},$

$$A = \{\log(x_{j_1} \cdots x_{j_n}) \mid j_k \in A_k, 1 \le k \le n\} \subset \mathbb{N}^n \text{ and}$$

$$B = \{\log(x_{j_1} \cdots x_{j_n}) \mid j_k \in B_k, 1 \le k \le n\} \subset \mathbb{N}^n.$$

It is easy to see that $A = \{\alpha \in \mathbb{N}^n \mid 0 \le \alpha_1 + \ldots + \alpha_{i_1} \le i_1 + 1 \text{ and } \mid \alpha \mid = n\}$ and
$B = \{\alpha \in \mathbb{N}^n \mid 0 \le \alpha_{t_2+1} + \ldots + \alpha_{t_2+i_2} \le i_2 + 1 \text{ and } \mid \alpha \mid = n\}$, if $i_2 + t_2 \le n$

or

$$B = \{\alpha \in \mathbb{N}^n \mid 0 \le \sum_{s=1}^{i_2+t_2-n} \alpha_s + \sum_{s=t_2+1}^{n} \alpha_s \le i_2 + 1 \text{ and } \mid \alpha \mid = n\}, \text{ if } i_2 + t_2 \ge n.$$

For any base ring $K[\mathcal{A}]$ of a transversal polymatroid presented by $\mathcal{A} = \{A_1, \ldots, A_n\}$ we associate a $(n \times n)$ square tiled by closed unit subsquares, called **boxes**, colored with two colors, "*white*" and "*black*", as follows: the box of coordinate (i, j) is "*white*" if $j \in A_i$, otherwise the box is "*black*". We will call this square the **polymatroidal diagram** associated to the presentation $\mathcal{A} = \{A_1, \ldots, A_n\}$.

Next we give necesary and sufficient conditions such that the $K -$ algebra $K[A \cap B]$ is the base ring associated to some transversal polymatroid.

Theorem 3.2.1. *Let* $1 \le i_1, i_2 \le n - 2$, $0 \le t_2 \le n - 1$. *We consider two presentations of transversal poymatroids presented by:* $\mathcal{A} = \{A_k \mid A_k = [n], \text{ if } k \in [i_1] \cup \{n\}, \; A_k = [n] \setminus [i_1], \text{ if } k \in [n-1] \setminus [i_1]\}$ *and* $\mathcal{B} = \{B_k \mid B_{\sigma^{t_2}(k)} = [n], \text{ if } k \in [i_2] \cup \{n\}, \; B_{\sigma^{t_2}(k)} = [n] \setminus \sigma^{t_2}[i_2], \text{ if } k \in [n-1] \setminus [i_2]\}$ *such that A, respectively B, is the set of exponent vectors of the monomials defining the base ring associated to the transversal polymatroid presented by $\mathcal{A}$, respectively $\mathcal{B}$.*
Then, the $K -$ algebra $K[A \cap B]$ is the base ring associated to a transversal polymatroid if and only if one of the following conditions holds:
a) $i_1 = 1$;
b) $i_1 \ge 2$ *and* $t_2 = 0$;
c) $i_1 \ge 2$ *and* $t_2 = i_1$;
d) $i_1 \ge 2$, $1 \le t_2 \le i_1 - 1$ *and* $i_2 \in \{1, \ldots, i_1 - t_2\} \cup \{n - t_2, \ldots, n - 2\}$;
e) $i_1 \ge 2$, $i_1 + 1 \le t_2 \le n - 1$ *and* $i_2 \in \{1, \ldots, n - t_2\} \cup \{n - t_2 + i_1, \ldots, n - 2\}$.

The proof follows from the following three lemmas.

Lemma 3.2.2. *Let A and B be as above. If $i_1 \ge 2$, $1 \le t_2 \le i_1 - 1$, then the $K -$ algebra*

$K[A \cap B]$ *is the base ring associated to some transversal polymatroid if and only if* $i_2 \in \{1, \ldots, i_1 - t_2\} \cup \{n - t_2, \ldots, n - 2\}$.
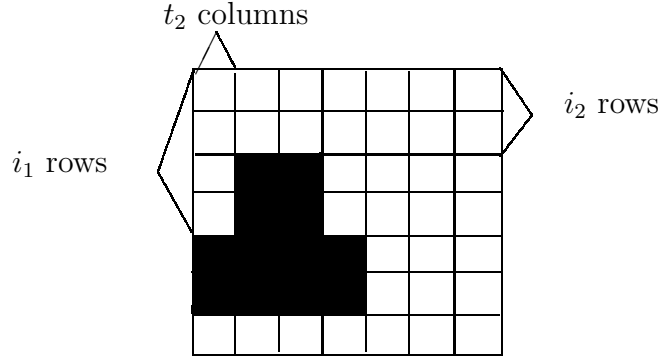
Proof. $"\Leftarrow"$ Let $i_2 \in \{1, \ldots, i_1 - t_2\} \cup \{n - t_2, \ldots, n - 2\}$. We will prove that there exists a transversal polymatroid $\mathcal{P}$ presented by $\mathcal{C} = \{C_1, \ldots, C_n\}$ such that the base ring associated to $\mathcal{P}$ is $K[A \cap B]$.

We have two cases to study.

Case 1. If $i_2 + t_2 \leq i_1$, then let $\mathcal{P}$ be the transversal polymatroid presented by $\mathcal{C} = \{C_1, \ldots, C_n\}$, where

$$C_1 = \ldots = C_{i_2} = C_n = [n],$$
$$C_{i_2+1} = \ldots = C_{i_1} = [n] \setminus \sigma^{t_2}[i_2],$$
$$C_{i_1+1} = \ldots = C_{n-1} = [n] \setminus [i_1].$$

The associated polymatroidal diagram is the following.

t_2 columns

i_2 rows

i_1 rows

It is easy to see that the base ring associated to the transversal polymatroid $\mathcal{P}$ presented by $\mathcal{C}$ is generated by the following set of monomials

$$\{x_{t_2+1}, \ldots, x_{t_2+i_2}\}^{i_2+1-k} \{x_1, \ldots, x_{t_2}, x_{t_2+i_2+1}, \ldots, x_{i_1}\}^{i_1-i_2+k-s} \{x_{i_1+1}, \ldots, x_n\}^{n-1-i_1+s}$$

for any $0 \leq k \leq i_2 + 1$ and $0 \leq s \leq i_1 - i_2 + k$. If $x^\alpha \in K[\mathcal{C}]$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, then there exist $0 \leq k \leq i_2 + 1$ and $0 \leq s \leq i_1 - i_2 + k$ such that

$$\alpha_{t_2+1} + \ldots + \alpha_{t_2+i_2} = i_2 + 1 - k \text{ and } \alpha_1 + \ldots + \alpha_{i_1} = i_1 + 1 - s$$

and thus, $K[\mathcal{C}] \subset K[A \cap B]$.

Conversely, if $\alpha \in A \cap B$ then $\alpha_{t_2+1} + \ldots + \alpha_{t_2+i_2} \leq i_2 + 1$ and $\alpha_1 + \ldots + \alpha_{i_1} \leq i_1 + 1$ and thus there exist $0 \leq k \leq i_2 + 1$ and $0 \leq s \leq i_1 - i_2 + k$ such that
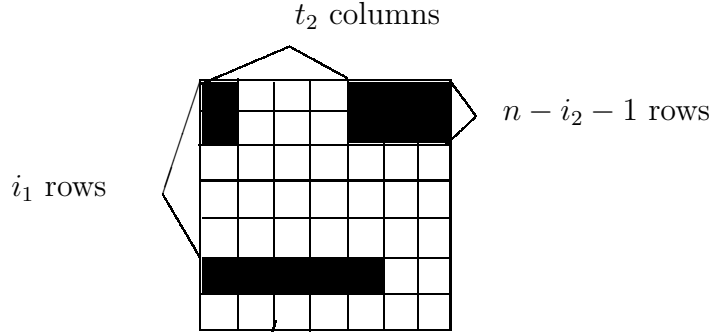
$$x^\alpha \in \{x_{t_2+1}, \ldots, x_{t_2+i_2}\}^{i_2+1-k}\{x_1, \ldots, x_{t_2}, x_{t_2+i_2+1}, \ldots, x_{i_1}\}^{i_1-i_2+k-s}\{x_{i_1+1}, \ldots, x_n\}^{n-1-i_1+s}.$$

Thus, $K[\mathcal{C}] \supset K[A \cap B]$ and so $K[\mathcal{C}] = K[A \cap B]$.

Case 2. If $i_2 + t_2 > i_1$, then it follows that $i_2 \geq n - t_2$ and $n - i_2 \leq t_2 < i_1$. Let $\mathcal{P}$ be the transversal polymatroid presented by $\mathcal{C} = \{C_1, \ldots, C_n\}$, where

$$
\begin{aligned}
C_1 &= \ldots = C_{n-i_2-1} = [n] \setminus \sigma^{t_2}[i_2], \\
C_{n-i_2} &= \ldots = C_{i_1} = C_n = [n], \\
C_{i_1+1} &= \ldots = C_{n-1} = [n] \setminus [i_1].
\end{aligned}
$$

The associated polymatroidal diagram is the following.

t_2 columns

$n - i_2 - 1$ rows

i_1 rows

It is easy to see that the base ring associated to the transversal polymatroid $\mathcal{P}$ presented by $\mathcal{C}$ is generated by the following set of monomials

$$\{x_{i_2+t_2-n+1}, \ldots, x_{t_2}\}^{i_1+1-k}\{x_1, \ldots, x_{i_2+t_2-n}, x_{t_2+1}, \ldots, x_{i_1}\}^{k-s}\{x_{i_1+1}, \ldots, x_n\}^{n-1-i_1+s}$$

for any $0 \leq k \leq i_1 + i_2 - n + 2$ and $0 \leq s \leq k$. Since $i_2 + t_2 \geq n$ and $0 \leq s \leq k \leq i_1 + i_2 - n + 2$, it follows that for any $x^\alpha \in K[\mathcal{C}]$ we have $\alpha_1 + \ldots + \alpha_{i_1} = i_1 + 1 - s \leq i_1 + 1$, $\alpha_1 + \ldots + \alpha_{i_2+t_2-n} + \alpha_{t_2+1} + \ldots + \alpha_n = n - 1 - i_1 + k \leq i_2 + 1$ and thus, $K[\mathcal{C}] \subset K[A \cap B]$.

Conversely, if $\alpha \in A \cap B$ then $\alpha_1 + \ldots + \alpha_{i_1} \leq i_1 + 1$, $\alpha_1 + \ldots + \alpha_{i_2 + t_2 - n} + \alpha_{t_2+1} + \ldots + \alpha_n \leq i_2 + 1$ and thus there exist $0 \leq k \leq i_1 + i_2 - n + 2$ and $0 \leq s \leq k$ such that

$$x^\alpha \in \{x_{i_2+t_2-n+1}, \ldots, x_{t_2}\}^{i_1+1-k}\{x_1, \ldots, x_{i_2+t_2-n}, x_{t_2+1}, \ldots, x_{i_1}\}^{k-s}\{x_{i_1+1}, \ldots, x_n\}^{n-1-i_1+s}.$$

Thus, $K[\mathcal{C}] \supset K[A \cap B]$ and so $K[\mathcal{C}] = K[A \cap B]$.

$"\Rightarrow"$ Now suppose that there exists a transversal polymatroid $\mathcal{P}$ given by $\mathcal{C} = \{C_1, \ldots, C_n\}$ such that its associated base ring is $K[A \cap B]$. We will prove that $i_2 \in \{1, \ldots, i_1 - t_2\} \cup \{n - t_2, \ldots, n - 2\}$.

Suppose, on the contrary, that $i_1 + 1 - t_2 \leq i_2 \leq n - t_2 - 1$. We have two cases to study:

Case $1'$. If $n - i_1 - 1 \leq i_2 + 1$, then since $(i_1+1)e_1 + (n-i_1-1)e_k \in \mathcal{P}$ for any $i_1 + 1 \leq k \leq n$ and $(i_1 + 1)e_1 + e_s + (n - i_1 - 2)e_k \notin \mathcal{P}$ for any $2 \leq s \leq i_1$ and $i_1 + 1 \leq k \leq n$, we may assume $1 \in C_1, \ldots, 1 \in C_{i_1}, 1 \in C_n$ and $C_{i_1+1} = \ldots = C_{n-1} = [n] \setminus [i_1]$. If $i_1 \leq i_2$, then since $(i_1 + 1)e_{t_2+1} + (n - i_1 - 1)e_{t_2+i_2+1} \in \mathcal{P}$, we may assume $t_2 + 1 \in C_1, \ldots, t_2 + 1 \in C_{i_1}, t_2 + 1 \in C_n$. Then $(i_1+1)e_{t_2+1} + (n-i_1-1)e_{i_1+1} \in \mathcal{P}$, which is false. If $i_1 > i_2$, then since $(i_2+1)e_{t_2+1} + (n-i_1-1)e_{t_2+i_2+1} \in \mathcal{P}$, we may assume $t_2 + 1 \in C_1, \ldots, t_2+1 \in C_{i_2}, t_2 + 1 \in C_n$. Then $(i_2 + 1)e_{t_2+1} + (i_1 - i_2)e_1 + (n - i_1 - 1)e_{i_1+1} \in \mathcal{P}$, which is false.

Case $2'$. If $n - i_1 - 1 > i_2 + 1$, then since $(i_1 + 1)e_1 + (i_2 + 1)e_{i_1+1} + (n - i_1 - i_2 - 2)e_k \in \mathcal{P}$ for any $t_2 + i_2 + 1 \leq k \leq n$ and $(i_1+1)e_1 + e_s + (i_2+1)e_{i_1+1} + (n-i_1-i_2-3)e_k \notin \mathcal{P}$ for any $1 \leq s \leq i_1$ and $t_2 + i_2 + 1 \leq k \leq n$, we may assume $1 \in C_1, \ldots, 1 \in C_{i_1}, 1 \in C_n, C_{i_1+1} = \ldots = C_{i_1+i_2+1} = [n] \setminus [i_1]$ and $C_{i_1+i_2+2} = \ldots = C_{n-1} = [n] \setminus [t_2 + i_2]$. If $i_1 \leq i_2$, then since $(i_1+1)e_{t_2+1} + (n-i_1-1)e_{t_2+i_2+1} \in \mathcal{P}$, we may assume $t_2+1 \in C_1, \ldots, t_2+1 \in C_{i_1}, t_2+1 \in C_n$. Then $(i_1+1)e_{t_2+1} + (i_2+1)e_{i_1+1} + (n-i_1-i_2-2)e_{t_2+i_2+1} \in \mathcal{P}$, which is false. If $i_1 > i_2$, then since $(i_2+1)e_{t_2+1} + (i_1-i_2)e_1 + (n-i_1-1)e_{t_2+i_2+1} \in \mathcal{P}$, we may assume $t_2+1 \in C_1, \ldots, t_2+1 \in C_{i_2}, t_2+1 \in C_n$. Then $(i_1-i_2)e_1 + (i_2+1)e_{t_2+1} + (i_2+1)e_{t_2+i_2} + (n-i_1-i_2-2)e_{t_2+i_2+1} \in \mathcal{P}$, which is false. $\square$

Lemma 3.2.3. *Let A and B be as above. If $i_1 \geq 2$, $i_1 + 1 \leq t_2 \leq n - 1$, then the $K - algebra$ $K[A \cap B]$ is the base ring associated to some transversal polymatroid if and only if $i_2 \in \{1, \ldots, n - t_2\} \cup \{n - t_2 + i_1, \ldots, n - 2\}$.*
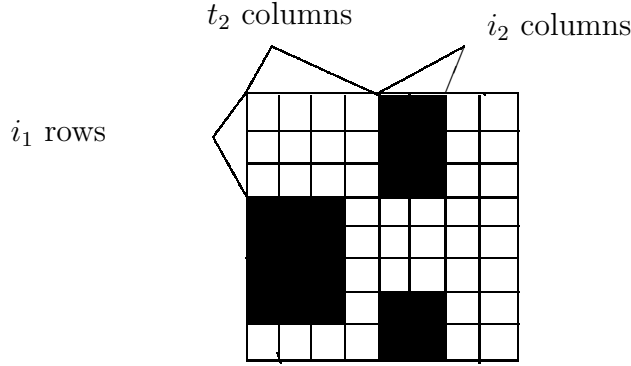
Proof. $"\Leftarrow"$ Let $i_2 \in \{1, \ldots, n - t_2\} \cup \{n - t_2 + i_1, \ldots, n - 2\}$. We will prove that there exists a transversal polymatroid $\mathcal{P}$ presented by $\mathcal{C} = \{C_1, \ldots, C_n\}$ such that its associated base ring is $K[A \cap B]$. We distinguish three cases to study:

Case 1. If $i_2 + t_2 \leq n$ and $i_1 + 1 + i_2 \neq n$, then let $\mathcal{P}$ be the transversal polymatroid

presented by $\mathcal{C} = \{C_1, \ldots, C_n\}$, where

$$C_1 = \ldots = C_{i_1} = C_n = [n] \setminus \sigma^{t_2}[i_2],$$
$$C_{i_1+1} = \ldots = C_{i_1+i_2+1} = [n] \setminus [i_1],$$
$$C_{i_1+i_2+2} = \ldots = C_{n-1} = [n] \setminus ([i_1] \cup \sigma^{t_2}[i_2]).$$

The associated polymatroidal diagram is the following.

t_2 columns i_2 columns

i_1 rows

It is easy to see that the base ring $K[\mathcal{C}]$ associated to the transversal polymatroid $\mathcal{P}$ presented by $\mathcal{C}$ is generated by the following set of monomials

$$\{x_1, \ldots, x_{i_1}\}^{i_1+1-k} \{x_{t_2+1}, \ldots, x_{t_2+i_2}\}^{i_2+1-s} \{x_{i_1+1}, \ldots, x_{t_2}, x_{t_2+i_2+1}, \ldots, x_n\}^{n-i_1-i_2-2+k+s}$$

for any $0 \leq k \leq i_1 + 1$ and $0 \leq s \leq i_2 + 1$. If $x^\alpha \in K[\mathcal{C}]$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, then there exist $0 \leq k \leq i_1 + 1$ and $0 \leq s \leq i_2 + 1$ such that

$$\alpha_{t_2+1} + \ldots + \alpha_{t_2+i_2} = i_2 + 1 - s , \quad \alpha_1 + \ldots + \alpha_{i_1} = i_1 + 1 - k$$

and thus, $K[\mathcal{C}] \subset K[A \cap B]$. Conversely, if $\alpha \in A \cap B$ then $\alpha_{t_2+1} + \ldots + \alpha_{t_2+i_2} \leq i_2 + 1$, $\alpha_1 + \ldots + \alpha_{i_1} \leq i_1 + 1$; thus there exist $0 \leq k \leq i_1 + 1$ and $0 \leq s \leq i_2 + 1$ such that

$$\alpha_{t_2+1} + \ldots + \alpha_{t_2+i_2} = i_2 + 1 - s , \quad \alpha_1 + \ldots + \alpha_{i_1} = i_1 + 1 - k$$
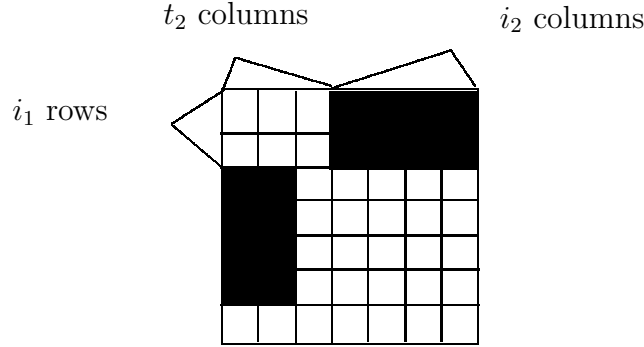
and since $\mid \alpha \mid = n$ it follows that

$$x^\alpha \in \{x_1, \ldots, x_{i_1}\}^{i_1+1-k} \{x_{t_2+1}, \ldots, x_{t_2+i_2}\}^{i_2+1-s} \{x_{i_1+1}, \ldots, x_{t_2}, x_{t_2+i_2+1}, \ldots, x_n\}^{n-i_1-i_2-2+k+s}.$$

Thus, $K[\mathcal{C}] \supset K[A \cap B]$ and so $K[\mathcal{C}] = K[A \cap B]$.

Case 2. If $i_2 + t_2 \leq n$ and $i_1 + 1 + i_2 = n$, then $t_2 = i_1 + 1$ and let $\mathcal{P}$ be the transversal polymatroid presented by $\mathcal{C} = \{C_1, \ldots, C_n\}$, where

$$C_1 = \ldots = C_{i_1} = [n] \setminus \sigma^{t_2}[i_2],$$
$$C_{i_1+1} = \ldots = C_{n-1} = [n] \setminus [i_1],$$
$$C_n = [n].$$

The associated polymatroidal diagram is the following.

t_2 columns i_2 columns

i_1 rows

It is easy to see that the base ring $K[\mathcal{C}]$ associated to the transversal polymatroid $\mathcal{P}$ presented by $\mathcal{C}$ is generated by the following set of monomials

$$\{x_1, \ldots, x_{i_1}\}^{i_1+1-k} x_{i_1+1}^{n-i_1-1+k-s} \{x_{i_1+2}, \ldots, x_n\}^s$$

for any $0 \leq k \leq i_1 + 1$ and $0 \leq s \leq n - i_1$. If $x^\alpha \in K[\mathcal{C}]$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, then there exist $0 \leq k \leq i_1 + 1$ and $0 \leq s \leq i_2 + 1 (= n - i_1)$ such that

$$\alpha_{t_2+1} + \ldots + \alpha_{t_2+i_2} = \alpha_{i_1+2} + \ldots + x_n = s \leq i_2 + 1 \ , \quad \alpha_1 + \ldots + \alpha_{i_1} = i_1 + 1 - k$$

and thus, $K[\mathcal{C}] \subset K[A \cap B]$. Conversely, if $\alpha \in A \cap B$ then $\alpha_{t_2+1} + \ldots + \alpha_{t_2+i_2} = \alpha_{i_1+2} + \ldots + x_n \leq i_2 + 1$ and $\alpha_1 + \ldots + \alpha_{i_1} \leq i_1 + 1$; thus there exist $0 \leq k \leq i_1 + 1$ and $0 \leq s \leq i_2 + 1$ such that

$$x^\alpha \in \{x_1, \ldots, x_{i_1}\}^{i_1+1-k} x_{i_1+1}^{n-i_1-1+k-s} \{x_{i_1+2}, \ldots, x_n\}^s.$$
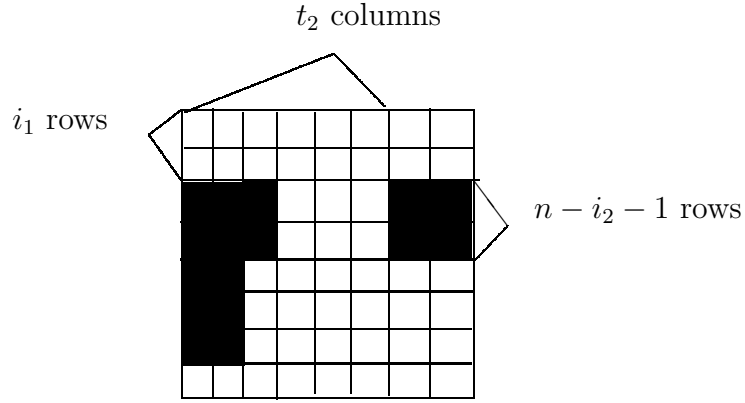
Thus, $K[\mathcal{C}] \supset K[A \cap B]$ and so $K[\mathcal{C}] = K[A \cap B]$.

Case 3. If $i_2 + t_2 > n$, then let $\mathcal{P}$ be the transversal polymatroid presented by $\mathcal{C} =$

$\{C_1, \ldots, C_n\}$, where

$$C_1 = \ldots = C_{i_1} = C_n = [n],$$
$$C_{i_1+1} = \ldots = C_{i_1+n-i_2-1} = [n] \setminus \sigma^{t_2}[i_2],$$
$$C_{i_1+n-i_2} = \ldots = C_{n-1} = [n] \setminus [i_1].$$

The associated polymatroidal diagram is the following:

t_2 columns

i_1 rows

$n - i_2 - 1$ rows

Since $i_2 + t_2 > n$, it follows that $i_2 + t_2 \geq n + i_1$ and so $i_2 - i_1 \geq n - t_2 \geq 1$. It is easy to see that the base ring $K[\mathcal{C}]$ associated to the transversal polymatroid $\mathcal{P}$ presented by $\mathcal{C}$ is generated by the following set of monomials

$$\{x_1, \ldots, x_{i_1}\}^{i_1+1-k} \{x_{i_1+1}, \ldots, x_{i_2+t_2-n}, x_{t_2+1}, \ldots, x_n\}^{i_2-i_1+k-s} \{x_{i_2+t_2-n+1}, \ldots, x_{t_2}\}^{n-i_2-1+s}$$

for any $0 \leq k \leq i_1 + 1$ and $0 \leq s \leq i_2 - i_1 + k$. If $x^\alpha \in K[\mathcal{C}]$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, then there exist $0 \leq k \leq i_1 + 1$ and $0 \leq s \leq i_2 - i_1 + k$ such that

$$\alpha_1 + \ldots + \alpha_{i_1} = i_1 + 1 - k , \quad \alpha_1 + \ldots + \alpha_{i_2+t_2-n} + \alpha_{t_2+1} + \ldots + \alpha_n = i_2 + 1 - s$$

and thus, $K[\mathcal{C}] \subset K[A \cap B]$. Conversely, if $\alpha \in A \cap B$ then $\alpha_1 + \ldots + \alpha_{i_1} \leq i_1 + 1$ and $\alpha_1 + \ldots + \alpha_{i_2+t_2-n} + \alpha_{t_2+1} + \ldots + \alpha_n \leq i_2 + 1$, then there exist $0 \leq k \leq i_1 + 1$ and $0 \leq s \leq i_2 - i_1 + k$

such that

$$\alpha_1 + \ldots + \alpha_{i_1} = i_1 + 1 - k \ , \quad \alpha_1 + \ldots + \alpha_{i_2+t_2-n} + \alpha_{t_2+1} + \ldots + \alpha_n = i_2 + 1 - s$$

and since $\mid \alpha \mid = n$ it follows that

$$x^\alpha \in \{x_1, \ldots, x_{i_1}\}^{i_1+1-k} \{x_{i_1+1}, \ldots, x_{i_2+t_2-n}, x_{t_2+1}, \ldots, x_n\}^{i_2-i_1+k-s} \{x_{i_2+t_2-n+1}, \ldots, x_{t_2}\}^{n-i_2-1+s}.$$

Thus, $K[\mathcal{C}] \supset K[A \cap B]$ and so $K[\mathcal{C}] = K[A \cap B]$.

$\quad " \Rightarrow "$ Now suppose that there exists a transversal polymatroid $\mathcal{P}$ presented by $\mathcal{C} = \{C_1, \ldots, C_n\}$ such that its associated base ring is $K[A \cap B]$. We will prove that $i_2 \in \{1, \ldots, n-t_2\} \cup \{n-t_2+i_1, \ldots, n-2\}$. Suppose, on the contrary, that $i_2 \in \{n-t_2+1, \ldots, n-t_2+i_1-1\}$. We may assume that $1 \notin C_{i_2+t_2-n+1}, \ldots, 1 \notin C_{i_1}, 1 \notin C_{i_1+1}, \ldots, 1 \notin C_{n-1}$. If $i_1 < i_2$, then $x_1^{i_1+1} x_{i_1+1}^{n-i_1-1} \in K[A \cap B]$. But $x_1^{i_1+1} x_{i_1+1}^{n-i_1-1} \notin K[\mathcal{C}]$ because the maximal power of x_1 in a minimal generator of $K[\mathcal{C}]$ is $\leq i_2 + t_2 - n + 1 \leq i_1$, which is false. If $i_1 \geq i_2$, then $x_1^{i_2+1} x_{i_1+1}^{n-i_2-1} \in K[A \cap B]$. But $x_1^{i_2+1} x_{i_1+1}^{n-i_2-1} \notin K[\mathcal{C}]$ because the maximal power of x_1 in a minimal generator of $K[\mathcal{C}]$ is $\leq i_2 + t_2 - n + 1 \leq i_2$, which is false. Thus, $i_2 \in \{1, \ldots, n-t_2\} \cup \{n-t_2+i_1, \ldots, n-2\}$. $\qquad \square$

Lemma 3.2.4. *Let A and B as above. If $i_1 = 1$ and $0 \leq t_2 \leq n-1$, or $i_1 \geq 2$ and $t_2 = i_1$, or $i_1 \geq 2$ and $t_2 = 0$, then the $K-$algebra $K[A \cap B]$ is the base ring associated to some transversal polymatroid.*

Proof. We have three cases to study:

Case 1. $i_1 = 1$ and $0 \leq t_2 \leq n-1$. Then we distinguish five subcases:

Subcase 1.*a*. If $t_2 = 0$, then we find a transversal polymatroid $\mathcal{P}$ like in the Subcase 3.*b*. when $i_1 = 1$.

Subcase 1.*b*. If $t_2 > 0$ and $t_2 + i_2 \leq n$ with $i_2 \neq n-2, n-3$, then let $\mathcal{P}$ be the transversal polymatroid presented by $\mathcal{C} = \{C_1, \ldots, C_n\}$, where

$$
\begin{aligned}
C_1 &= C_n = [n] \setminus \sigma^{t_2}[i_2], \\
C_2 &= \ldots = C_{i_2+2} = [n] \setminus [1], \\
C_{i_2+3} &= \ldots = C_{n-1} = [n] \setminus \{\{1\} \cup \sigma^{t_2}[i_2]\}.
\end{aligned}
$$

It is easy to see that the polymatroid $\mathcal{P}$ is the same as in Lemma 3.2.3. when $i_2 + t_2 \leq n$ and $i_1 + 1 + i_2 \neq n$. Thus $K[A \cap B] = K[\mathcal{C}]$.

Subcase 1.c. If $t_2 > 0$ and $t_2 + i_2 \leq n$ with $i_2 = n - 2$, then let $\mathcal{P}$ be the transversal polymatroid presented by $\mathcal{C} = \{C_1, \ldots, C_n\}$, where

$$C_1 = [n] \setminus \sigma^{t_2}[n - 2] \;,\; C_n = [n],$$
$$C_2 = \ldots = C_{n-1} = [n] \setminus [1].$$

It is easy to see that the polymatroid $\mathcal{P}$ is the same as in Lemma 3.2.3. when $i_2 + t_2 \leq n$ and $i_1 + 1 + i_2 = n$. Thus $K[A \cap B] = K[\mathcal{C}]$.

Subcase 1.d. If $t_2 > 0$ and $t_2 + i_2 \leq n$ with $i_2 = n - 3$, then let $\mathcal{P}$ be the transversal polymatroid presented by $\mathcal{C} = \{C_1, \ldots, C_n\}$, where

$$C_1 = C_n = [n] \setminus \sigma^{t_2}[n - 3],$$
$$C_2 = \ldots = C_{n-1} = [n] \setminus [1].$$

It is easy to see that the polymatroid $\mathcal{P}$ is the same as in Lemma 3.2.3. when $i_2 + t_2 \leq n$ and $i_1 + 1 + i_2 \neq n$. Thus $K[A \cap B] = K[\mathcal{C}]$.

Subcase 1.e. If $t_2 > 0$ and $t_2 + i_2 > n$, then let $\mathcal{P}$ be the transversal polymatroid presented by $\mathcal{C} = \{C_1, \ldots, C_n\}$, where

$$C_1 = C_n = [n],$$
$$C_2 = \ldots = C_{n-i_2} = [n] \setminus \sigma^{t_2}[i_2],$$
$$C_{n-i_2+1} = \ldots = C_{n-1} = [n] \setminus \{1\}.$$

It is easy to see that the polymatroid $\mathcal{P}$ is the same as in Lemma 3.2.3. when $i_2 + t_2 > n$ and $i_1 = 1$. Thus $K[A \cap B] = K[\mathcal{C}]$.

Case 2. $i_1 \geq 2$ and $t_2 = i_1$. Then we distinguish three subcases:

Subcase 2.a. If $i_2 + t_2 < n - 1$, then let $\mathcal{P}$ be the transversal polymatroid presented by $\mathcal{C} = \{C_1, \ldots, C_n\}$, where

$$C_1 = \ldots = C_{i_1} = C_n = [n] \setminus \sigma^{t_2}[i_2],$$
$$C_{i_1+1} = \ldots = C_{i_1+i_2+1} = [n] \setminus [i_1],$$
$$C_{i_1+i_2+2} = \ldots = C_{n-1} = [n] \setminus [i_1 + i_2].$$

It is easy to see that the polymatroid $\mathcal{P}$ is the same as in Lemma 3.2.3. when $i_2 + t_2 \leq n$ and $i_1 + 1 + i_2 \neq n$. Thus $K[A \cap B] = K[\mathcal{C}]$.

Subcase 2.b. If $i_2 + t_2 = n - 1$, then let $\mathcal{P}$ be the transversal polymatroid presented by $\mathcal{C} = \{C_1, \ldots, C_n\}$, where

$$C_1 = \ldots = C_{i_1} = [n] \setminus \sigma^{t_2}[i_2],$$
$$C_{i_1+1} = \ldots = C_{n-1} = [n] \setminus [i_1],$$
$$C_n = [n].$$

It is easy to see that the polymatroid $\mathcal{P}$ is the same as in Lemma 3.2.3. when $i_2 + t_2 \leq n$ and $i_1 + 1 + i_2 = n$. Thus $K[A \cap B] = K[\mathcal{C}]$.

Subcase 2.c. If $i_2 + t_2 \geq n$, then let $\mathcal{P}$ be the transversal polymatroid presented by $\mathcal{C} = \{C_1, \ldots, C_n\}$, where

$$C_1 = \ldots = C_{n-i_2-1} = [n] \setminus \sigma^{t_2}[i_2],$$
$$C_{n-i_2} = \ldots = C_{i_1} = C_n = [n],$$
$$C_{i_1+1} = \ldots = C_{n-1} = [n] \setminus [i_1].$$

It is easy to see that the polymatroid $\mathcal{P}$ is the same as in Lemma 3.2.2. when $i_2 + t_2 > i_1$. Thus $K[A \cap B] = K[\mathcal{C}]$.

Case 3. $i_1 \geq 2$ and $t_2 = 0$. Then we distinguish two subcases:

Subcase 3.a. If $i_2 \leq i_1$, then let $\mathcal{P}$ be the transversal polymatroid presented by $\mathcal{C} = \{C_1, \ldots, C_n\}$, where
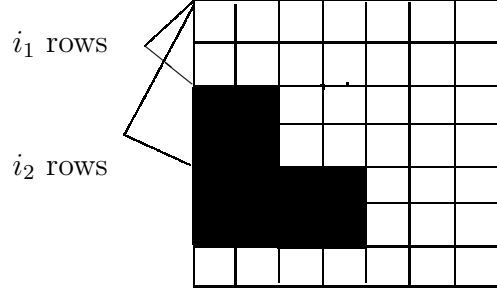
$$C_1 = \ldots = C_{i_2} = C_n = [n],$$
$$C_{i_2+1} = \ldots = C_{i_1} = [n] \setminus [i_2],$$
$$C_{i_1+1} = \ldots = C_{n-1} = [n] \setminus [i_1].$$

It is easy to see that the polymatroid $\mathcal{P}$ is the same as in Lemma 3.2.2. when $i_2 + t_2 \leq i_1$. Thus $K[A \cap B] = K[\mathcal{C}]$.

Subcase 3.b. If $i_2 > i_1$, then let $\mathcal{P}$ be the transversal polymatroid presented by $\mathcal{C} = \{C_1, \ldots, C_n\}$, where

$$C_1 = \ldots = C_{i_1} = C_n = [n],$$
$$C_{i_1+1} = \ldots = C_{i_2} = [n] \setminus [i_1],$$
$$C_{i_2+1} = \ldots = C_{n-1} = [n] \setminus [i_2].$$

The associated polymatroidal diagram is the following.

i_1 rows

i_2 rows

It is easy to see that the base ring $K[\mathcal{C}]$ associated to the transversal polymatroid $\mathcal{P}$ presented by $\mathcal{C}$ is generated by the following set of monomials

$$\{x_1, \ldots, x_{i_1}\}^{i_1+1-k}\{x_{i_1+1}, \ldots, x_{i_2}\}^{i_2-i_1+k-s}\{x_{i_2+1}, \ldots, x_n\}^{n-i_2+s-1}$$

for any $0 \leq k \leq i_1 + 1$ and $0 \leq s \leq i_2 - i_1 + k$. If $x^\alpha \in K[\mathcal{C}]$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, then there exist $0 \leq k \leq i_1 + 1$ and $0 \leq s \leq i_2 - i_1 + k$ such that

$$\alpha_1 + \ldots + \alpha_{i_2} = i_2 + 1 - s \text{ and } \alpha_1 + \ldots + \alpha_{i_1} = i_1 + 1 - k$$

and thus, $K[\mathcal{C}] \subset K[A \cap B]$. Conversely, if $\alpha \in A \cap B$ then $\alpha_1 + \ldots + \alpha_{i_2} \leq i_2 + 1$ and $\alpha_1 + \ldots + \alpha_{i_1} \leq i_1 + 1$ and so there exist $0 \leq k \leq i_1 + 1$ and $0 \leq s \leq i_2 - i_1 + k$ such that

$$\alpha_1 + \ldots + \alpha_{i_2} = i_2 + 1 - s \text{ and } \alpha_1 + \ldots + \alpha_{i_1} = i_1 + 1 - k$$

and since $\mid \alpha \mid = n$ it follows that

$$\{x_1, \ldots, x_{i_1}\}^{i_1+1-k}\{x_{i_1+1}, \ldots, x_{i_2}\}^{i_2-i_1+k-s}\{x_{i_2+1}, \ldots, x_n\}^{n-i_2+s-1}.$$

Thus, $K[\mathcal{C}] \supset K[A \cap B]$ and so $K[\mathcal{C}] = K[A \cap B]$.

$\square$

Chapter 4

A remark on the Hilbert series of transversal polymatroids

In this chapter we study when the transversal polymatroids presented by $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, with all sets A_i having two elements, have the base ring $K[\mathcal{A}]$ Gorenstein. Using Worpitzky identity, we prove that the numerator of Hilbert series has the coefficients Eulerian numbers and from [1] it follows that the Hilbert series is unimodal.

4.1 Segre product and the base ring associated to a transversal polymatroid.

Let K be an infinite field, n and m be positive integers, A_i be some subsets of $[n]$ for $1 \leq i \leq m$, $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$. Let

$$K[\mathcal{A}] = K[x_{i_1} x_{i_2} \ldots x_{i_m} \mid i_j \in A_j, 1 \leq j \leq m]$$

and

$$C = K[x_i y_j \mid i \in A_j, 1 \leq j \leq m].$$

Obviously $C \subseteq S$, where S is the Segre product of polynomial rings in n, respectively m, indeterminates

$$S := K[x_1, x_2, \ldots, x_n] * K[y_1, y_2, \ldots, y_m] = K[x_i y_j \mid 1 \leq i \leq n, 1 \leq j \leq m].$$

We consider the variables t_{ij}, $1 \leq i \leq n$, $1 \leq j \leq m$, and we define

$$T = K[t_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m],$$

$$T(\mathcal{A}) = K[t_{ij} \mid 1 \leq j \leq m, i \in A_j].$$

We also consider the presentations $\phi : T \longrightarrow S$ and $\phi^{'} : T(\mathcal{A}) \longrightarrow C$ defined by $t_{ij} \longrightarrow x_i y_j$. By [28, Proposition 9.1.2] we know that $\ker(\phi)$ is the ideal $I_2(t)$ of the 2–minors of the $n \times m$ matrix $t = (t_{ij})$ via the map ϕ. The algebras C, $T(\mathcal{A})$, S and T are $\mathbb{Z}^m$ -graded by setting $\deg(x_i y_j) = \deg(t_{ij}) = e_j \in \mathbb{Z}^m$ where e_j, $1 \leq j \leq m$, denote the vectors of the canonical basis of $\mathbb{Z}^m$. By [20, Propositions 4.11 and 8.11] or [28, Proposition 8.1.10] we know that the cycles of the complete bipartite graph $K_{n,m}$ give a universal Gröbner basis of $I_2(t)$. A cycle of the complete bipartite graph is described by a pair (I, J) of sequences of integers, say

$$I = i_1, i_2, \ldots, i_s, \ \ J = j_1, j_2, \ldots, j_s,$$

with $2 \leq s \leq \min(m, n)$, $1 \leq i_k \leq m$, $1 \leq j_k \leq n$, and such that the i_k are distinct and the j_k are distinct. Associated with any such a pair we have a polynomial $F_{(I,J)} = t_{i_1 j_1} \ldots t_{i_s j_s} - t_{i_2 j_1} \ldots t_{i_s j_{s-1}} t_{i_1 j_s}$ which is in $I_2(t)$.

For a $\mathbb{Z}^m$-graded algebra E we denote by E_Δ the direct sum of the graded components of degree $(a, a, \ldots, a) \in \mathbb{Z}^m$. Similarly, for a $\mathbb{Z}^m-$graded $E-$module M, we denote by M_Δ the direct sum of the graded components of M of degree $(a, a, \ldots, a) \in \mathbb{Z}^m$. Clearly E_Δ is a $\mathbb{Z}-$graded algebra and M_Δ is a $\mathbb{Z}-$graded E_Δ module. Furthermore $-_\Delta$ is exact as a functor on the category of $\mathbb{Z}^m-$ graded $E-$modules with maps of degree 0. Now C_Δ is the K-algebra generated by the elements $x_{i_1} y_1 \ldots x_{i_m} y_m$ with $i_j \in A_j$. Therefore $K[\mathcal{A}]$ is isomorphic to the algebra C_Δ and we have the presentation

$$0 \longrightarrow J \longrightarrow T(\mathcal{A})_\Delta \longrightarrow K[\mathcal{A}] \longrightarrow 0,$$

where $J = I_2(t) \cap T(\mathcal{A})_\Delta$.

$T(\mathcal{A})_\Delta$ is the $K-$algebra generated by the monomials $t_{1 i_1} t_{2 i_2} \ldots t_{m i_m}$, with $i_k \in A_k$, that is, $T(\mathcal{A})_\Delta$ is the Segre product $T_1 * T_2 * \ldots * T_m$ of the polynomial rings $T_i = K[t_{ij} \mid j \in A_i]$. Now we consider the variables s_α with $\alpha \in A := A_1 \times A_2 \times \ldots \times A_m$. Then we get the presentation of the Segre product $T(\mathcal{A})_\Delta$ as a quotient of $K[s_\alpha \mid \alpha \in A]$ by mapping $s_{(j_1, \ldots, j_m)}$ to $t_{1 j_1} t_{2 j_2} \ldots t_{m j_m}$.

From [15] the defining ideal of $T(\mathcal{A})_\Delta$ is generated by the so-called Hibi relations

$$s_\alpha s_\beta - s_{(\alpha \vee \beta)} s_{(\alpha \wedge \beta)},$$

where

$$\alpha \vee \beta = (\max(\alpha_1, \beta_1), \dots, \max(\alpha_m, \beta_m)),$$

and

$$\alpha \wedge \beta = (\min(\alpha_1, \beta_1), \dots, \min(\alpha_m, \beta_m)).$$

Example 4.1.1. Let $n = 3$ and $\mathcal{A} = \{A_1 = \{1, 2\}, A_2 = \{2, 3\}, A_3 = \{3, 4\}\}$. Then C is the quotient of $K[t_{11}, t_{12}, t_{22}, t_{23}, t_{33}, t_{34}]$ by zero ideal ($J = 0$ because we don't have cycles) and then $K[\mathcal{A}] = K[x_1 x_2 x_3, x_1 x_2 x_4, x_1 x_3^2, x_1 x_3 x_4, x_2^2 x_3, x_2^2 x_4, x_2 x_3^2, x_2 x_3 x_4]$ is the quotient of $K[s_{123}, s_{124}, s_{133}, s_{134}, s_{223}, s_{224}, s_{233}, s_{234}]$ modulo the ideal generated by the Hibi relations:

$$s_{123} s_{134} - s_{124} s_{133} \ , \ s_{123} s_{224} - s_{124} s_{223},$$

$$s_{123} s_{234} - s_{124} s_{233} \ , \ s_{123} s_{233} - s_{133} s_{223},$$

$$s_{123} s_{234} - s_{133} s_{224} \ , \ s_{123} s_{234} - s_{134} s_{223},$$

$$s_{124} s_{234} - s_{134} s_{224} \ , \ s_{133} s_{234} - s_{134} s_{233},$$

$$s_{223} s_{234} - s_{224} s_{233}.$$

Since $K[t_{11}, t_{12}] * K[t_{22}, t_{23}] * K[t_{33}, t_{34}]$ is a Gorenstein ring ([16, Example 7.4]), B_3 is a Gorenstein ring .

Example 4.1.2. Let $n = 3$ and $\mathcal{A} = \{A_1 = \{1, 2\}, A_2 = \{2, 3\}, A_3 = \{3, 1\}\}$. Then C is the quotient of $K[t_{11}, t_{12}, t_{22}, t_{23}, t_{33}, t_{31}]$ by the polynomial $t_{11} t_{22} t_{33} - t_{12} t_{23} t_{31}$ (we have one 6-cycle) and then $K[\mathcal{A}] = K[x_1 x_2 x_3, x_1^2 x_2, x_1 x_3^2, x_1^2 x_3, x_2^2 x_3, x_1 x_2^2, x_2 x_3^2]$ is the quotient of $K[s_{123}, s_{121}, s_{133}, s_{131}, s_{223}, s_{221}, s_{233}, s_{231}]$ modulo the ideal generated by the Hibi relations:

$$s_{221} s_{233} - s_{223} s_{231}, \ s_{131} s_{233} - s_{133} s_{231},$$

$$s_{121} s_{233} - s_{123} s_{231}, \ s_{131} s_{221} - s_{121} s_{231},$$

$$s_{133} s_{221} - s_{123} s_{231}, \ s_{131} s_{223} - s_{123} s_{231},$$

$$s_{133} s_{223} - s_{233} s_{231}, \ s_{121} s_{223} - s_{221} s_{231},$$

$$s_{121} s_{133} - s_{131} s_{231},$$

and by the linear relation

$$s_{123} - s_{231}.$$

Since $K[t_{11}, t_{12}] * K[t_{22}, t_{23}] * K[t_{33}, t_{31}]$ is a Gorenstein ring and $t_{11} t_{22} t_{33} - t_{12} t_{23} t_{31}$ is a regular element in $K[t_{11}, t_{12}] * K[t_{22}, t_{23}] * K[t_{33}, t_{31}]$, $\frac{K[t_{11}, t_{12}] * K[t_{22}, t_{23}] * K[t_{33}, t_{31}]}{(t_{11} t_{22} t_{33} - t_{12} t_{23} t_{31})} \cong K[\mathcal{A}]$ is a Gorenstein ring.

4.2 Hilbert series

Definition 4.2.1. Let $R = K[x_1, x_2, \ldots, x_n]$ be a polynomial ring over a field K. If M is a finitely generated $\mathbb{N}-$graded R-module, the numerical function

$$H(M, -) : \mathbb{N} \longrightarrow \mathbb{N}$$

with $H(M, n) = \dim_K(M_n)$, for all $n \in \mathbb{N}$, is the Hilbert function and

$$H_M(t) = \sum_{n \in \mathbb{N}} H(M, n) t^n$$

is the Hilbert series of M.

Definition 4.2.2. A sequence $(h_i)_{i \geq 0}$ is *log-concave* if $h_i^2 \geq h_{i-1} h_{i+1}$ for all $i \geq 1$.

Definition 4.2.3. A sequence $(h_i)_{i \geq 0}$ is *unimodal* if there exists an index $j \geq 2$ such that $h_i \leq h_{i+1}$ for $i \leq j - 1$ and $h_i \geq h_{i+1}$ for $i \geq j$.

Log-concavity is easily shown to imply unimodality.

Let n, m be positive integers, A_i be some subsets of $[n]$ such that $|A_i| = l$ for $1 \leq i \leq m$, $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$,

$$B_m = K[x_{i_1} x_{i_2} \ldots x_{i_m} \mid i_j \in A_j, 1 \leq j \leq m]$$

and

$$C = K[x_i y_j \mid i \in A_j, 1 \leq j \leq m].$$

From section above we know that B_m is isomorphic to the algebra C_Δ and we have the presentation

$$0 \longrightarrow J \longrightarrow T(\mathcal{A})_\Delta \longrightarrow B_m \longrightarrow 0,$$

where $J = I_2(t) \cap T(\mathcal{A})_\Delta$.

Now we are interested when $J = (0)$.

Remark 4.2.4. If $J = (0)$ then B_m is isomorphic to the algebra $T(\mathcal{A})_\Delta$.

The ideal J is zero if and only if when the bipartite graph presented by $\mathcal{A}$ ($V_1 = \{1, 2, \ldots, m\}, V_2 = A_1 \cup A_2 \cup \ldots \cup A_m$ and the edges from V_1 to V_2 are the following: join $i \in V_1$ with $i_j \in V_2 \Leftrightarrow i_j \in A_i$) has no cycles. If $|A_i| = l$ for $1 \leq i \leq m$, $|A_i \cap A_{i+1}| \leq 1$

and $A_j \cap A_i = \varnothing$ for $2 \leq i \leq m$, $j < i - 1$ then the bipartite graph presented by $\mathcal{A}$ has no cycles , thus the ideal J is zero.

Since $J = (0)$, then B_m is the Segre product of m polynomial rings, each of them in l indeterminates, that is, B_m is a Gorenstein ring (see [16, Example 7.4]); $\dim_K(B_m)_i = \binom{i+l-1}{i}^m$. In the case $m = 2$ it is known (see [28, proposition 9.1.3]) that the Hilbert series of B_2 is

$$H_{B_2}(t) = \frac{\sum_{k=0}^{l-1} \binom{l-1}{k}^2 t^k}{(1-t)^{2l-1}}; H(B_2, i) = \dim_K(B_2)_i = \binom{i+l-1}{i}^2.$$

It results that the Krull dimension of B_2 is $\dim_K B_2 = 2l - 1$ and the number of generators of the defining ideal of B_2 (the number of Hibi-relations of B_2) is

$$\mu = \binom{H(B_2, 1) + 1}{2} - H(B_2, 2) = \binom{l^2 + 1}{2} - \binom{l+1}{2}^2 = \binom{l}{2}^2.$$

Proposition 4.2.5. *We have the following relation between the Hilbert series of B_{m+1} and B_m*

$$H_{B_{m+1}}(t) = \frac{1}{(l-1)!} \frac{d^{(l-1)}}{dt^{l-1}} (t^{l-1} H_{B_m}(t)).$$

Proof. Since

$$H_{B_m}(t) = \sum_{i \geq 0} \binom{i+l-1}{i}^m t^i, \text{ we have}$$

$$\frac{1}{(l-1)!} \frac{d^{(l-1)}}{dt^{l-1}} (t^{l-1} H_{B_m}(t)) = \frac{1}{(l-1)!} \frac{d^{(l-1)}}{dt^{l-1}} (t^{l-1} \sum_{i \geq 0} \binom{i+l-1}{i}^m t^i)$$

$$= \frac{1}{(l-1)!} \frac{d^{(l-2)}}{dt^{l-2}} (\frac{d}{dt} (t^{l-1} \sum_{i \geq 0} \binom{i+l-1}{i}^m t^i))$$

$$= \frac{1}{(l-1)!} \frac{d^{(l-2)}}{dt^{l-2}} ((l-1)t^{l-2} \sum_{i \geq 0} \binom{i+l-1}{i}^m t^i + t^{l-2} \sum_{i \geq 0} i \binom{i+l-1}{i}^m t^i)$$

$$= \frac{1}{(l-1)!} \frac{d^{(l-2)}}{dt^{l-2}} (t^{l-2} \sum_{i \geq 0} \binom{i+l-1}{i}^m (i+l-1)t^i)$$

$$= \frac{1}{(l-1)!} \frac{d^{(l-3)}}{dt^{l-3}} (\frac{d}{dt} (t^{l-2} \sum_{i \geq 0} \binom{i+l-1}{i}^m (i+l-1)t^i))$$

$$= \frac{1}{(l-1)!} \frac{d^{(l-3)}}{dt^{l-3}} ((l-2)t^{l-3} \sum_{i \geq 0} \binom{i+l-1}{i}^m (i+l-1)t^i +$$

$$+ t^{l-3} \sum_{i \geq 0} i \binom{i+l-1}{i}^m (i+l-1)t^i)$$

$$= \frac{1}{(l-1)!} \frac{d^{(l-3)}}{dt^{l-3}} (t^{l-3} \sum_{i \geq 0} \binom{i+l-1}{i}^m (i+l-1)(i+l-2)t^i) = \cdots$$

$$= \frac{1}{(l-1)!} \sum_{i \geq 0} \binom{i+l-1}{i}^m (i+l-1)(i+l-2) \cdots (i+2)(i+1)t^i$$

$$= \sum_{i \geq 0} \binom{i+l-1}{i}^{m+1} t^i = H_{B_{m+1}}(t).$$

$\square$

Definition 4.2.6. ([12]) Let $A(t) := \sum_i a_i t^i$ and $B(t) := \sum_i b_i t^i$ be two power series in $\mathbb{Z}[[t]]$. Then we define the Hadamard product of A and B and we denote it by $Had(A, B) := \sum_i (a_i b_i) t^i$.

Definition 4.2.7. ([12]) Let $A(t)$ be the Hilbert series of a standard $K-$algebra S. Then we denote by $ri(A)$(or $ri(S)$) the regularity index of A (or of S), i.e. the first integer r such that for every $s \geq r$ the Hilbert function of S takes the same values as the Hilbert polynomial of S.

Remark 4.2.8. $ri(S) = a(S) + 1$, where $a(S)$ is the $a-$invariant of S.

Proposition 4.2.9. ([12]) Let $A(t) := \frac{P(t)}{(1-t)^a}$ and $B(t) := \frac{Q(t)}{(1-t)^b}$, where $p := deg(P)$, $q := deg(Q)$, $P(1) \neq 0$, $Q(1) \neq 0$, and assume that $A(t)$ and $B(t)$ are the Hilbert series of standard $K-$algebras. Then:

1) $ri(A) = p - a + 1$ and $ri(B) = q - b + 1$;

2) $ri(Had(A, B)) \leq \max(ri(A), ri(B))$;

3) $Had(A, B) = \frac{R(t)}{(1-t)^{a+b-1}}$ with $R(1) \neq 0$;

4) $deg(R) \leq \max(ri(A), ri(B)) + (a + b - 1) - 1$.

Theorem 4.2.10. ([12]) Let S_1 and S_2 be two standard $K-$algebras and assume that we know their Hilbert series, H_{S_1} and H_{S_2}. Then the Hilbert series of the Segre product of S_1 and S_2 is $H_{S_1 * S_2} = Had(H_{S_1}, H_{S_2})$.

Definition 4.2.11. Let $R = K[x_1, x_2, \ldots, x_n]$ be a polynomial ring over a field K, M is a finitely generated $\mathbb{N}-$ graded R-module. The *difference operator* Δ on the set of numerical functions $H(M, -)$ is

$$(\Delta H(M, -))(n) = H(M, n + 1) - H(M, n),$$

where $H(M, -)$ is the Hilbert function of M.

The $m - times$ $iterated$ Δ $operator$ $("m - difference$ of $H(M, n)")$ will be denoted by Δ^m.

Proposition 4.2.12. *If* $|A_i| = 2$ *for* $1 \leq i \leq m$, $|A_i \cap A_{i+1}| \leq 1$ *and* $A_j \cap A_i = \varnothing$ *for* $1 \leq i \leq m - 1$, $1 \leq j < i - 1$ *then the Hilbert series of B_m is*

$$H_{B_m}(t) = \frac{\sum_{k=0}^{m-1} A(m, k+1)t^k}{(1 - t)^{m+1}},$$

where

$$A(m, k) = kA(m - 1, k) + (m - k + 1)A(m - 1, k - 1),$$

with $A(m, 1) = A(m, m) = 1$ *and* $2 \leq k \leq m - 1$.

Proof. We know that $B_m = T_1 * T_2 * \ldots * T_m$, where $T_i = K[t_{ij} \mid j \in A_i]$ is the Segre product of m polynomial rings in two indeterminates and $\dim_K(B_m)_i = \binom{i+2-1}{i}^m = (i + 1)^m$.

We will show that B_m has Krull dimension $\dim B_m = m + 1$ and the Hilbert series, $H_{B_m}(t) = \frac{R(t)}{(1-t)^{m+1}}$ with $\deg(R) \leq m - 1$.
We proceed by induction on $m \geq 1$. If $m = 1$ it is clear. Suppose $m \geq 2$. For every $1 \leq i \leq m$ we have $ri(H_{T_i}) = -1$, thus $ri(H_{B_m}) = -1$. Since $B_{m+1} = B_m * T_{m+1}$, we have

$$H_{B_{m+1}}(t) = Had(H_{B_m}, T_{m+1}) = \frac{R(t)}{(1 - t)^{(m+1)+2-1}} = \frac{R(t)}{(1 - t)^{m+2}};$$

$$\deg(R) \leq \max(ri(H_{B_m}), ri(H_{T_{m+1}})) + ((m + 1) + 2 - 1) - 1 = m.$$

Now we will find the coefficients $r_i's$ of the Hilbert series $H_{B_m}(t) = \frac{R(t)}{(1-t)^{m+1}}$, where $R(t) := \sum_{k=0}^{m-1} r_i t^i$. We may compute the first m values of $H(B_m, i)$. Then it suffices to take the $(m + 1)^{st}$ difference of these first m values and we get the required $r_i's$. For this it suffices to go backward in the algorithm which determines the numerators of the Hilbert series and to obtain $H(B_m, i) = dim_K(B_m)_i$ for all i.

We define
$$A_0(m, k) = r_k = A(m, k),$$

$$A_i(m, 1) = 1, A_i(m, k) = A_i(m, k - 1) + A_{i-1}(m, k),$$

for $i \geq 1$ and $2 \leq k \leq m$.

For $m \geq 2$ and $2 \leq k \leq m$ fixed we want to prove that

$$A_t(m, k) = \sum_{s=1}^{k} A(m, s) \binom{t + k - s - 1}{k - s}$$

for any $t \geq 1$.

We proceed by induction on $t \geq 1$.

Case $t = 1$. Since for any $m \geq 2$ and $2 \leq k \leq m$ fixed we have

$$A_1(m, k) = A_1(m, k - 1) + A(m, k),$$
$$A_1(m, k - 1) = A_1(m, k - 2) + A(m, k - 1),$$
$$A_1(m, k - 2) = A_1(m, k - 3) + A(m, k - 2),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$A_1(m, 3) = A_1(m, 2) + A(m, 3),$$
$$A_1(m, 2) = A_1(m, 1) + A(m, 2),$$
$$A_1(m, 1) = 1 = A(m, 1),$$

we obtain

$$A_1(m, k) = \sum_{s=1}^{k} A(m, s).$$

Case $t > 1$.

From

$$A_t(m, k + 1) = A_t(m, k) + A_{t-1}(m, k + 1),$$
$$A_{t-1}(m, k + 1) = A_{t-1}(m, k) + A_{t-2}(m, k + 1),$$
$$A_{t-2}(m, k + 1) = A_{t-2}(m, k) + A_{t-3}(m, k + 1),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_3(m, k+1) = A_3(m, k) + A_2(m, k+1),$$
$$A_2(m, k+1) = A_2(m, k) + A_1(m, k+1),$$
$$A_1(m, k+1) = A_1(m, k) + A(m, k+1),$$

we obtain

$$A_t(m, k+1) = \sum_{j=1}^{t} A_j(m, k) + A(m, k+1).$$

For $t > 1$

$$A_t(m, k+1) = \sum_{j=1}^{t} A_j(m, k) + A(m, k+1)$$

$$= \sum_{j=1}^{t} \left(\sum_{s=1}^{k} A(m, s) \binom{j+k-s-1}{k-s} \right) + A(m, k+1)$$

$$= \sum_{s=1}^{k} \left(\sum_{j=1}^{t} \binom{j+k-s-1}{k-s} \right) A(m, s) + A(m, k+1)$$

$$= \sum_{s=1}^{k} \binom{t+k-s}{k-s+1} A(m, s) + A(m, k+1)$$

$$= \sum_{s=1}^{k+1} A(m, s) \binom{t+k-s}{k-s+1},$$

since

$$\sum_{j=1}^{t} \binom{j+k-s-1}{k-s} = \binom{t+k-s}{k-s+1}.$$

Now we want to prove that $A_{m+1}(m, k) = k^m$.

From [11] or [35] we mention the Worpitzky identity

$$k^m = \sum_{s=1}^{m} A(m, s) \binom{k+s-1}{m}.$$

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We know that

$$A_{m+1}(m,k) = \sum_{s=1}^{k} A(m,s)\binom{m+k-s}{k-s} = \sum_{s=1}^{k} A(m,s)\binom{m+k-s}{m}.$$

Thus

$$k^m = \sum_{s=1}^{m} A(m,s)\binom{k+s-1}{m} = A(m,m)\binom{k+m-1}{m} + A(m,m-1)$$
$$\times \binom{k+m-1-1}{m} + \ldots + A(m,m-k+2)\binom{m+1}{m} + A(m,m-k+1)\binom{m}{m}$$
$$= A(m,1)\binom{k+m-1}{m} + A(m,2)\binom{k+m-2}{m} + \ldots + A(m,k-1)\binom{k+m-k+1}{m}$$
$$+ A(m,k)\binom{k+m-k}{m} = \sum_{s=1}^{k} A(m,s)\binom{m+k-s}{m} = A_{m+1}(m,k).$$

Thus we have $r_k = A(m,k+1)$ for $0 \le k \le m-1$. $\qquad\square$

Corollary 4.2.13. *The sequence in k, $A(m,k)$ with $1 \le k \le m$, is symmetric for any $m \ge 2$.*

Proof. If $m = 2$ then $A(2,1)=A(2,2)=1$.
If $m > 2$ then $A(m,k) = k\, A(m-1,k) + (m-k+1)\, A(m-1,k-1) = kA(m-1,m-k) + (m-k+1)A(m-1,m-k+1) = A(m,m-k+1)$. $\qquad\square$

Corollary 4.2.14. *The number of generators of the defining ideal of B_m (the number of Hibi$-$ relations of B_m) is*

$$\mu = \binom{H(B_m,1)+1}{2} - H(B_m,2) = \binom{2^m+1}{2} - 3^m = 2^{2m-1} + 2^{m-1} - 3^m.$$

Corollary 4.2.15. *The h-vector of the Hilbert series associated to the transversal polymatroid presented by $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, such that $|A_i| = 2$ for $1 \le i \le m$, $|A_i \cap A_{i+1}| \le 1$ and $A_j \cap A_i = \varnothing$ for $1 \le i \le m-1$, $1 \le j < i-1$, is unimodal .*

Proof. From [1], we know that $A(m,k)$ is a log-concave sequence in k, for all m, thus it is unimodal. $\qquad\square$

Bibliography

[1] M.Bona, R.Ehrenborg, *Combinatorial Proof of the Log-Concavity of the Numbers of Permutations with k Runs*, Journal of Combinatorial Theory, Series A 90, (2000), 293–303.

[2] A. Brøndsted, *Introduction to Convex Polytopes*, Graduate Texts in Mathematics 90, Springer-Verlag, 1983.

[3] W. Bruns, J. Herzog, *Cohen-Macaulay rings,* Revised Edition, Cambridge, 1997.

[4] W. Bruns, J. Gubeladze, *Polytopes, rings and K-theory,* preprint.

[5] W. Bruns, R. Koch, *Normaliz–a program for computing normalizations of affine semigroups*, 1998. Available via anonymous ftp from: ftp.mathematik.Uni-Osnabrueck.DE/pub/osm/kommalg/software.

[6] A. Conca, *Linear Spaces, Transversal Polymatroids and ASL Domains*, Journal of Algebraic Combinatorics, Volume 25, Issue 1, (2007) 25–41.

[7] D. Eisenbud *Commutative algebra with a view toward algebraic geometry*, Springer, 1994.

[8] G.-M.Greuel, G.Pfister, H.Schönemann. SINGULAR 2.0. *A Computer Algebra System for Polynomial Computations*. Centre for Computer Algebra, University of Kaiserslautern(2001). http://www.singular.uni-kl.de.

[9] T. Hibi, *Algebraic Combinatorics on Convex Polytopes*, Carslaw Publications, Glebe, N.S.W., Australia, 1992.

[10] J. Edmonds, *Submodular functions, matroids, and certain polyedra*, in *Combinatorial Structures and Their Applications*, (R. Guy, H. Hanani, N. Sauer, J. Schonheim, Eds.), Gordon and Breach, New York, 1970.

[11] L. Comtet. *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, rev. enl. ed. Dordrecht, Netherlands: Reidel, 1974.

[12] L. Robbiano, G. Valla, *Hilbert-Poincaré Series of Bigraded Algebra,* Bollettino U.M.I.(8), (1998), 521–540.

[13] V. Reiner and V. Welker, *On the Charney-Davis and Neggers-Stanley conjectures.* Avaible at www.math.umn.edu/ reiner.

[14] J. Oxley, *Matroid Theory*, Oxford University Press, Oxford, 1992.

[15] T. Hibi, *Algebraic Combinatorics on Convex Polytopes*, Carslaw Publication, Glebe, N.S.W., Australia, 1992.

[16] J. Herzog, T. Hibi, *Discrete polymatroids*, J. Algebraic Combin., 16(2002), no. 3, 239–268.

[17] E. De Negri, T Hibi, *Gorenstein algebras of Veronese type*, J. Algebra 193, (1997), 629–639.

[18] J. Herzog, T. Hibi, M. Vlădoiu, *Ideals of fiber type and polymatroids*, Osaka J. Math. 42 (2005), 807–829.

[19] R. P. Stanley, *Combinatorics and Commutative Algebra*, Birkhäuser, 1983.

[20] B. Sturmfels, *Gröbner bases and convex polytopes,* Amer. Math. Soc., 1996.

[21] A. Ştefan, *A class of transversal polymatroids with Gorenstein base ring,* Bull. Math. Soc. Sci. Math. Roumanie Tome 51(99) No. 1, (2008), 67–79.

[22] A. Ştefan, *Intersections of base rings associated to transversal polymatroids*, to appear in Bull. Math. Soc. Sci. Math. Roumanie Tome 51(99) No. 4, (2008), http://arXiv.org/pdf/:0805.2729.

[23] A. Ştefan, *The type of the base ring associated to a transversal polymatroid*, submitted, http://arXiv.org/pdf/:0807.2371.

[24] A. Ştefan, *A remark on the Hilbert series of transversal polymatroids* , Analele Ştiinţifice ale Universitaţii "Ovidius" Constanţa Seria Matematica volumul XIV (2006), fascicola 2, 85–96.

[25] A. Ştefan, *The Facets Cone Associated To Some Classes Of Transversal Polymatroids*, Analele Ştiinţifice ale Universitaţii "Ovidius" Constanţa Seria Matematica volumul XV (2007), fascicola 1, 139–158.

[26] A. Ştefan, *Some examples of transversal polymatroids with Gorenstein base ring*, Preprint 2006.

[27] E. Miller, B. Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics 227, Springer-Verlag, New-York, 2005.

[28] R. Villarreal, *Monomial Algebras,* Marcel Dekker, New-York, 2001.

[29] M. Vlădoiu, *Discrete polymatroids*, An. Şt. Univ. Ovidius, Constanţa, 14 (2006), 89–112.

[30] M. Vlădoiu, *Equidimensional and unmixed ideals of Veronese type*, to appear in Communications in Alg., arXiv:math. AC/0611326.

[31] G. M. Ziegler, *Lecture on Polytopes*, Graduate Texts in Mathematics 152, Springer-Verlag, New-York, 1995.

[32] R. Webster, *Convexity*, Oxford University Press, Oxford, 1994.

[33] D. Welsh, *Matroid Theory*, Academic Press, London, 1976.

[34] N. White, *The basis monomial ring of a matroid*, Adv. in Math. 24(1977) 292–294.

[35] J. Worpitzky, *Studienüber die Bernoullischen und Eulerschen Zahlen.* J. reine angew. Math. 94, (1883), 203–232.

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